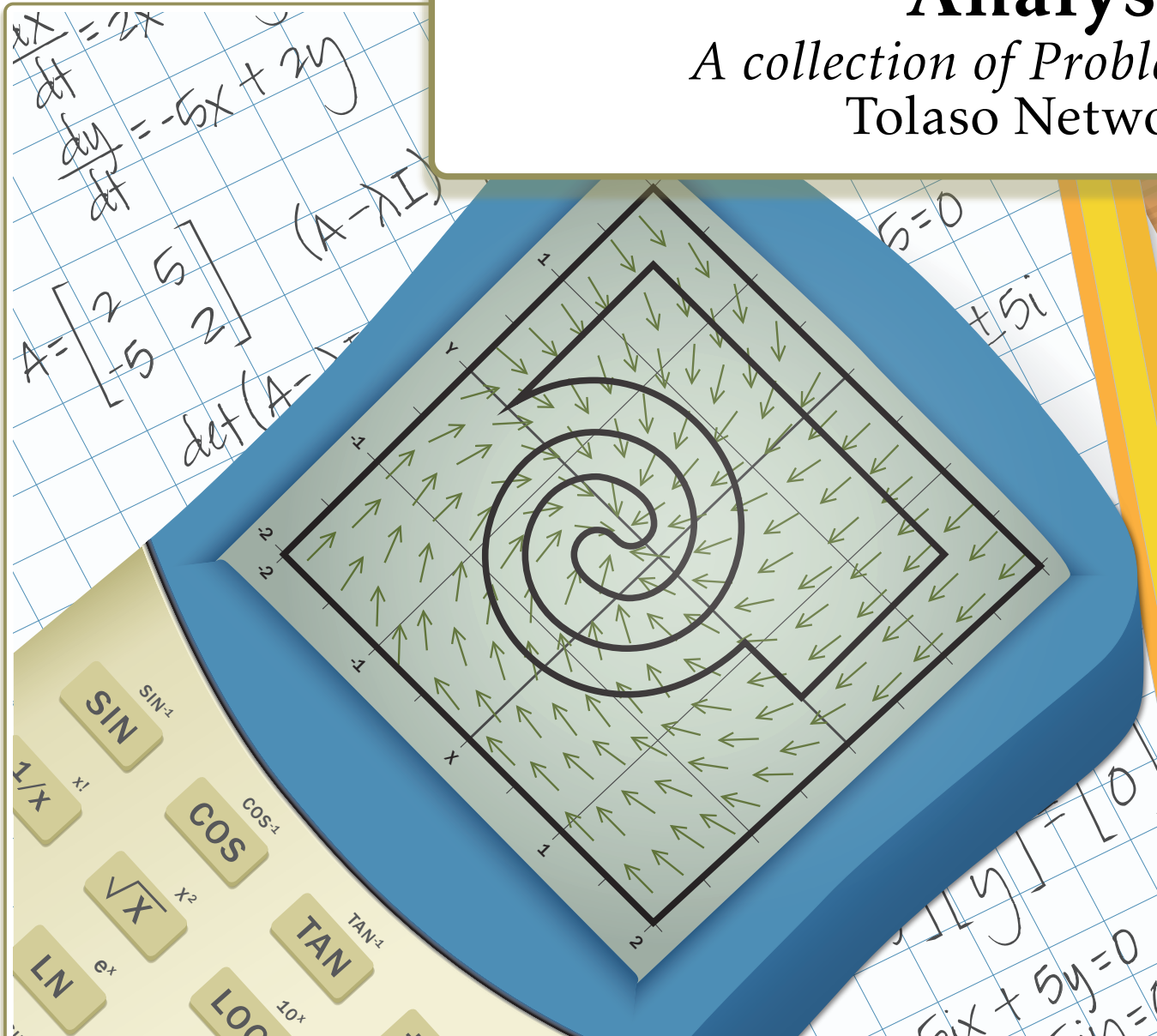# Mathematical Analysis
A collection of Problems
Tolaso Network

MATHEMATICS

November 2018

MATHEMATICAL ANALYSIS

A collection of problems

Social Media and Email Account

Joy of Mathematics   Tolaso Network   tolaso@tolaso.com.gr   Donate Tolaso Network

Mini Contents

Foreword

Dear reader,

there are a lot of interesting analysis problems scattered in the Internet World. Navigating through different sites you may encounter an exercise that will catch your attention and possibly you may want to archive it in your collection so to have access to it later. This is the main idea behind this booklet. The attempt started back in 2014 when an effort to collect as many exercises as possible began. Basic ideas are being recycled frequently and reappear in many exercises although unrelated at first.

The booklet contains a collection of interesting problems in Mathematical Analysis. The problems come from various branches of mathematics.

- ◆ Real and Complex Analysis
- ◆ General Topology
- ◆ Multivariable Calculus
- ◆ Integrals and Series

In each section the reader of this booklet shall encounter exercises that may find out there. Many of them are known to you but still they are interesting. However , there do exist exercises that demand creativity in order to be solved. The level of difficulty varies from exercise to exercise and in no way are the problems ordered according to their level of difficulty.

The author (Tolaso) started the collection of the problems using exercises that he encountered in his university classes (Calculus I, Calculus II and Calculus IV) and found to be the most interesting and fascinating. He dediced to include non trivial problems (as these have nothing to offer usually and rely mostly on definitions) but challenging ones.

The version you are now reading is Version 11 which is an improvement of the previous Version 10. I would like to personally thank all those people who contacted me personally to mention any typographical and / or mathematical errors that were corrected in this version. A big thanks to all of you guys ! I am open to your e-mails for improvements / suggestions . Feel free to contact me at the e-mail address that you will find at page 2. Last but not least , you are free to use the booklet as an instructive tutorial to your students. However , be very careful when assigning exercises to them.

Tolaso J Kos

November 27, 2018

Acknowledgements

- ♟ Many thanks to all those people (from all around the world) who embraced this booklet and have sent remarks and / or suggestions so that it is improved as well as selecting some of its exercises to assign to their students. I really appreciate it.
- ♟ The people at TeX Stack Exchange who have suggested some hacks for some parts of the existing code so that everything fits within the specified margins as well as the suggestion for the first page.

Donation

If you like the work done for this booklet as well as the overall work produced by Tolaso Network and want to donate please follow the link found at page 2. We thank you in advance.

Real - Complex Analysis

(1) For which $a \in \mathbb{R}$ does the sequence

$$\gamma_n = (1 + a)(1 + 2a^2) \cdots (1 + na^n)$$

converge? Give a brief explanation.

(2) We define a sequence x_n as follows

$$x_{n+3} = \frac{x_{n+2}^2 + 5x_{n+1}^2 + x_n^2}{x_{n+2} + 5x_{n+1} + x_n}$$

where $x_1, x_2, x_3 > 0$. Examine whether the sequence converges.

(3) A sequence of real number $\{x_n\}_{n \in \mathbb{N}}$ satisfies the condition

$$|x_n - x_m| > \frac{1}{n} \quad \text{whenever} \quad n < m$$

Prove that x_n is not bounded.

(4) Prove that

$$\lim_{n \to +\infty} \left((n+1)^{(n+2)/(n+1)} - n^{(n+1)/n}\right) = 1$$

(5) Prove that

$$\lim_{n \to +\infty} n \sin(2\pi e n!) = 2\pi$$

(6) Prove that the limit

$$\ell = \lim_{n \to +\infty} \frac{\tan n}{n}$$

does not exist.

(7) Find the value of

$$\ell = \sqrt{6 + \sqrt{6 + \sqrt{6 + \sqrt{\cdots}}}}$$

(8) Let $\lfloor \cdot \rfloor$ denote the floor function. Define

$$a_n = \sqrt{n} - \lfloor \sqrt{n} \rfloor$$

    (a) Prove that the limit points of a_n is the set $[0, 1]$.

    (b) Prove that $\limsup a_n = 1$.

(9) Let $\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}$ and $\{y_n\}_{n=1}^{\infty} \subset (0, +\infty)$. Suppose that $\{x_n/y_n\}_{n=1}^{\infty}$ is monotone. Prove that the sequence $\{z_n\}_{n \in \mathbb{N}}$ defined as

$$z_n = \frac{x_1 + x_2 + \cdots + x_n}{y_1 + y_2 + \cdots + y_n}$$

is also monotone.

(10) Evaluate the limit

$$\ell = \lim_{n \to +\infty} \frac{n + n^2 + n^3 + \cdots + n^n}{1^n + 2^n + 3^n + \cdots + n^n}$$

(11) Let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence defined as

$$x_n = \sin 1 + \sin 3 + \sin 5 + \cdots + \sin(2n - 1)$$

Find the supremum as well as the infimum of the sequence x_n.

(12) Let $\alpha \in \mathbb{R}$ such that $\alpha/\pi \notin \mathbb{Q}$. Prove that the sequence

$$\omega_n = \sin(\sin \alpha) + \sin(\sin(2\alpha)) + \cdots + \sin(\sin(n\alpha))$$

is bounded.

(13) Let $\{a_n\}_{n \in \mathbb{N}}$ be a sequence of real numbers that is defined recursively as

$$a_{n+1} = \sqrt{a_n a_{n-1}} \quad n \geqslant 2$$

and $a_2 > a_1 > 0$.

    (a) Prove that a_n converges.

    (b) Prove that $\lim_{n \to +\infty} a_n = \sqrt[3]{a_2^2 a_1}$.

    (c) Let $\mathcal{D}_n$ be a closed interval with endpoints the terms a_{2n}, a_{2n-1} of the sequence. Find the intersection of the intervals $\mathcal{D}_n$, $n \in \mathbb{N}$.

(14) Let $0 < a < b$ be real numbers. Define a sequence x_n as follows $x_1 = a$, $x_2 = b$ and

$$x_{2n+1} = \sqrt{x_{2n} x_{2n-1}}, \quad x_{2n+2} = \frac{x_{2n} + x_{2n-1}}{2}$$

Prove that the sequence converges and find its limit. 🖥

🖥The limit of the sequence is

$$\ell = \frac{2\sqrt{x_2}\sqrt{x_2 - x_1}}{2\log\left(\sqrt{x_2} + \sqrt{x_2 - x_1}\right) - \log x_1}$$

(15) For a sequence $A = (a_0, a_1, a_2, \dots)$ of reals, let
$SA = (a_0, a_0 + a_1, a_0 + a_1 + a_2, \dots)$ be the sequence
of its partial sums $a_0 + a_1 + a_2 + \dots$. Can one find a
non-zero sequence A for which the sequences
$A, SA, SSA, SSSA, \dots$ are all convergent?

(16) Let F_n denote the n-th Fermat number $2^{2^n} + 1$.
Evaluate the limit

$$\ell = \lim_{n \to +\infty} \sqrt{6F_1 + \sqrt{6F_2 + \sqrt{6F_3 + \sqrt{\dots + \sqrt{6F_n}}}}}$$

🍃

(17) Let $\mathcal{H}_n$ denote the n - th harmonic number. Define the
sequence a_n as follows

$$a_n = n^{\mathcal{H}_n \, \mathrm{lcm}(1,2,\dots,n)}$$

where $\mathrm{lcm}(, \cdot,)$ is the least common multiple.

  (a) Prove that $\log a_n \sim e^n \log^2 n$. 🍃

  (b) Prove that all the terms of the sequence are
     integers.

(18) Define a sequence a_n as follows

$$a_1 = \frac{1}{2} \quad , \quad a_{n+1} = \frac{1 + a_n^2}{2}$$

🍃

  (a) Use induction to prove that $\frac{1}{2} \leqslant a_n < a_{n+1} < 1$.

  (b) Prove that the sequence converges and find its
     limit.

  (c) Prove that for $n \geqslant 9$ it holds that

$$\left| a_n - \frac{n}{n+1} \right| > \left| a_n - \frac{n-1}{n+1} \right|$$

(19) Define

$$f_n(x) = \frac{x^n}{n!} \quad , \quad x \in \mathbb{R} \, , \, n \in \mathbb{N}$$

Examine the pointwise convergence as well as the
uniform convergence of f_n.

🍃$\ell = \frac{13}{2}$

🍃In fact this is equivelant to proving $\mathrm{lcm}(1, 2, \dots, n) \sim e^n$.

🍃This is called *quadratic map*. For more information check at
http://mathworld.wolfram.com/QuadraticMap.html

(20) Given the sequence of functions

$$f_n(x) = \cos^n x, \ 0 \leqslant x \leqslant \pi$$

Prove that

  (a) $\lim f_n(x) = 0$ but $f_n(\pi)$ does not converge.

  (b) Prove that f_n converges pointwise but not
     uniformly on $[0, \pi/2]$.

(21) Let $f : [0, 1] \to [0, 1]$ be a continuous function. We
define

$$x_{n+1} = f(x_n) \quad , \quad x_0 \in [0, 1]$$

where x_0 is picked arbitrary. If $x_{n+1} - x_n \to 0$, then
prove that x_n converges.

(22) Let $\{a_n\}_{n \in \mathbb{N}}$ be a real valued sequence such that the
series $\sum\limits_{n=1}^{\infty} a_n^2$ converges. Prove that the series $\sum\limits_{n=1}^{\infty} \frac{a_n}{n}$
also converges.

(23) Let $\{a_n\}_{n \in \mathbb{N}}$ be a positive real valued sequence. If the
series $\sum\limits_{n=1}^{\infty} a_n$ converges prove that the series
$\sum\limits_{n=1}^{\infty} a_n^{n/(n+1)}$ also converges.

(24) Let u_n be a sequence such that

$$\left| \frac{u_{n+1}}{u_n} \right| = 1 + \frac{A}{n} + \mathcal{O}\left(\frac{1}{n^2} \right)$$

where A does not depend on n and $A < -1$. Prove that
the seies $\sum\limits_{n=1}^{\infty} u_n$ converges absolutely.

(25) Let $\alpha \in \mathbb{R} \setminus \mathbb{Z}$ and let us denote with $\lfloor \cdot \rfloor$ the floor
function. Prove that the series

$$S = \sum_{n=1}^{\infty} \left(\alpha - \frac{\lfloor n\alpha \rfloor}{n} \right)$$

diverges.

(16th Cuban Mathematical Olympiad)

(26) Let a_n be a positive and strictly decreasing sequence
such that $\lim a_n = 0$. Prove that the series

$$S = \sum_{n=1}^{\infty} \frac{a_n - a_{n+1}}{a_n}$$

diverges. ≋

(27) Let $\mathbb{P}$ denote the set of prime numnbers. Discuss the convergence of the series

$$S = \sum_{p \in \mathbb{P}} \frac{\sin p}{p}$$

(28) Examine whether the (double) series

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\sin(\sin(nm))}{n^2 + m^2}$$

converges. ≋

(29) Let $\{X_n\}_{n \in \mathbb{N}}$ be a sequence of strictly increasing positive integers. For each $n \geqslant 1$ let W_n be the least common multiple of the first n terms $X_1, X_2, \ldots, X_n$. Prove that , as $n \to +\infty$, the series

$$S = \frac{1}{W_1} + \frac{1}{W_2} + \cdots + \frac{1}{W_n}$$

converges.

(30) Let $\{a_n\}_{n \in \mathbb{N}}$ be a strictly increasing sequence of positive integers. Prove that the series $\displaystyle\sum_{i=0}^{n} \frac{1}{[a_i, a_{i+1}]}$ converges. Here $[\cdot, \cdot]$ denotes the least common multiple. ≋

≋**Hint:** Let $x_1, \ldots, x_n \in (0, 1)$. It holds that

$$\sum_{i=1}^{n}(1 - x_i) \geqslant 1 - \prod_{i=1}^{n} x_i$$

≋It appears that this problem is quite difficult. It appeared in several fora including math.stackexchange.com as well as mathematica.gr. In both went answered till today. In math.stackexchange.com they suggest that the series converges and its limit is $\frac{1}{2}$.
≋**Hint:**

$$\sum_{i=0}^{n} \frac{1}{[a_i, a_{i+1}]} = \sum_{i=0}^{n} \frac{(a_i, a_{i+1})}{a_i a_{i+1}}$$
$$\leqslant \sum_{i=0}^{n} \frac{a_{i+1} - a_i}{a_i a_{i+1}}$$
$$= \sum_{i=0}^{n} \frac{1}{a_i} - \frac{1}{a_{i+1}}$$

(31) Let $f : (0, +\infty) \to \mathbb{R}$ be a positive differentiable function such that its derivative is positive. Prove that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{f(n)}$ converges if-f the series

$$\sum_{n=1}^{\infty} \frac{f^{-1}(n)}{n^2} \text{ converges.}$$

(32) Let $\mathcal{H}_n$ denote the n-th harmonic number. Study the convergence of the series

$$S = \sum_{n=1}^{\infty} \alpha^{\mathcal{H}_n}$$

for the different values of $\alpha > 0$.

(33) Let $\mathcal{H}_n$ denote the n -th harmonic number. Prove that the series

$$S = \sum_{n=1}^{\infty} (-1)^n \frac{\sqrt[n]{\log n!}}{\log(\mathcal{H}_{n+1})}$$

converges.

(34) Let $\mathcal{H}_n$ denote the n - th harmonic number. Prove that the series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\log(\mathcal{H}_n)}{e^{\mathcal{H}_n}}$$

converges.

(35) Let $\mathcal{H}_n$ denote the n - th harmonic number. Prove that the series

$$\sum_{n=1}^{\infty} \frac{n^{\mathcal{H}_n}}{(\mathcal{H}_n)^n}$$

converges.

(36) Let $\{a_n\}_{n \in \mathbb{N}}$ be a positive real valued sequence such that the series $\displaystyle\sum_{n=1}^{\infty} a_n$ converges. Examine the convergence of the series

$$S = \sum_{n=1}^{\infty} \left(1 - \frac{\sin a_n}{a_n}\right)$$

$$= \frac{1}{a_0} - \frac{1}{a_n} < \frac{1}{a_0}$$

37 Let $\{x_n\}_{n \in \mathbb{N}}$ be a real valued sequence of positive terms such that $\sum\limits_{n=1}^{\infty} x_n$ converges. Set

$$s_n = \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}$$

Prove that the series $\sum\limits_{n=1}^{\infty} \frac{n^2}{x_n s_n^2}$ converges.

38 Let $\alpha \in \mathbb{R}$. For which values of α does the series

$$S = \sum_{n=1}^{\infty} \left(\frac{\pi}{2} - \arcsin \frac{n}{n+4} \right)^{\alpha}$$

converge?

39 Examine the convergence of the series

$$S = \sum_{n=1}^{\infty} \frac{\sin(\sin n)}{n}$$

Does it converge absolutely? Justify your answer.

40 Let a_n be a sequence of positive terms and suppose that $\sum\limits_{n=1}^{\infty} a_n$ converges.

(a) Prove that the series $\sum\limits_{n=1}^{\infty} \frac{n}{\sum_{k=1}^{n} a_k}$ also converges.

(b) Find the smallest possible value of λ such that

$$\sum_{n=1}^{\infty} \frac{n}{\sum_{k=1}^{n} a_k} \leqslant \lambda \sum_{n=1}^{\infty} \frac{1}{a_n}$$

41 Prove that the series

$$S_{\alpha} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{\alpha}} \sin(\log n)$$

converges if and only if $\alpha > 0$.

42 For what values of $x \in \mathbb{R}$ do the series

(i) $S_1 = \sum\limits_{n=1}^{\infty} \cos(2^n x)$ (ii) $S_2 = \sum\limits_{n=1}^{\infty} \sin(2^n x)$

converge?

43 Define x_n recursively as:

$$x_1 = 1 \quad , \quad x_{n+1} = \sin x_n$$

(a) Prove that $x_n \sim \sqrt{\frac{3}{n}}$.

(b) Prove that x_n converges to 0 monotonically decreasing.

(c) What inequality should β satisfy in order the series

$$S = \sum_{n=1}^{\infty} x_n^{\beta}$$

to converge?

44 Let $a_n \sim \mathrm{Bern}\left(\frac{1}{2}\right)$ i.e. each a_n is 0 or 1 with probability $\frac{1}{2}$. Prove that the series

$$S = \sum_{n=1}^{\infty} \frac{a_n}{n}$$

is almost everywhere divergent.

45 What can you say about the uniform convergence of the series

$$S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(n\pi x), \quad x \in \mathbb{R}$$

46 Let $x \in \mathbb{R}$. Consider the series

$$S = \sum_{n=2}^{\infty} \frac{\sin nx}{\log n} \qquad (1)$$

(A) (a) Prove that S converges for all $x \in \mathbb{R}$.

  (b) Prove that (1) is not a Fourier series of a Lebesgue integrable function.

(B) Examine if the function defined at (1) is continuous. Give a brief explanation to support your argument.

(C) Prove that the series $\sum\limits_{n=2}^{\infty} \frac{\cos nx}{\log n}$ is both Riemann and Lebesgue integrable as well as a Fourier series.

Hint: It holds that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sin n\pi x = \begin{cases} \dfrac{\pi x}{2} & , \quad 0 \leqslant x < 1 \\ 0 & , \quad x = 1 \\ \dfrac{\pi(x-2)}{2} & , \quad 1 < x \leqslant 2 \end{cases}$$

Do the same question for the quite similar series $\sum\limits_{n=2}^{\infty} \frac{\sin nx}{n \log n}$.

(47) Let $a \in \mathbb{Z}$. Define the function

$$f(x) = \sin ax, \ x \in (0, \pi)$$

Prove that f can be expanded into a Fourier cosine series and that it holds

$$\sin ax \sim \begin{cases} \dfrac{4a}{\pi} \displaystyle\sum_{n=0}^{\infty} \dfrac{\cos(2n+1)x}{a^2 - (2n+1)^2} &, \ a \text{ even} \\ \dfrac{4a}{\pi} \left[\dfrac{1}{2a^2} + \displaystyle\sum_{n=1}^{\infty} \dfrac{\cos 2nx}{a^2 - 4n^2} \right] &, \ a \text{ odd} \end{cases}$$

(48) Let $A \subseteq \mathbb{R}$ be a set of finite measure.

  (a) Find the Fourier series of $|\sin \lambda x|$.

  (b) Evaluate the limit

$$\ell = \lim_{\lambda \to +\infty} \int_A |\sin \lambda x| \, dx$$

(49) Let $f : [-\pi, \pi] \to \mathbb{R}$ be defined as $f(x) = |x|$.

  (a) Expand f in a Fourier series.

  (b) Prove that

    (i) $\displaystyle\sum_{n=1}^{\infty} \dfrac{1}{n^2} = \dfrac{\pi^2}{6}$

    (ii) $\displaystyle\sum_{n=1}^{\infty} \dfrac{1}{(2n-1)^2} = \dfrac{\pi^2}{8}$

  (c) Apply Parseval's identity to evaluate the series

$$S = \sum_{n=1}^{\infty} \dfrac{1}{n^4}$$

(50) Expand the function

$$f(x) = \log(1 - \cos x) \ , \ x \in \left(0, \dfrac{\pi}{2}\right)$$

in a Fourier series.

(51) Let $\{a_n\}_{n \in \mathbb{N}}$ be a bounded sequence. Prove that the sequence of functions defined as $\displaystyle\sum_{n=1}^{\infty} \dfrac{a_n}{n^{2x}}$ converges absolutely and uniformly on $(0, +\infty)$ to a differentiable function.

(Question from a Real Analysis Exam

University of Ioannina , Greece)

(52) Examine if there exists an $1 - 1$ function $f : \mathbb{N} \to \mathbb{N}$ such that $\displaystyle\sum_{n=1}^{\infty} \dfrac{f(n)}{n^2}$ converges.

(53) Examine whether the series

$$S = \sum_{n=1}^{\infty} \sin\left[\pi \left(2 + \sqrt{3}\right)^n\right]$$

converges.

(54) Examine whether the series

$$S = \sum_{n=1}^{\infty} \left(e - \left(1 + \tfrac{1}{n}\right)^n \right)$$

converges.

(55) Let $\{a_n\}_{n \in \mathbb{N}}$ be a real valued sequence such that the series $\displaystyle\sum_{n=1}^{\infty} \dfrac{a_n}{n}$ converges. Prove that

$$\lim_{n \to +\infty} \dfrac{a_1 + a_2 + \cdots + a_n}{n} = 0$$

(56) Given the sequence of $f_n : \mathbb{R} \to \mathbb{R}$ where $n \in \mathbb{N}$ defined as

$$f_n(x) = \sum_{n=1}^{\infty} \dfrac{n}{n^3 + x^2}$$

prove that

  (a) the serieses $\displaystyle\sum_{n=1}^{\infty} f_n$ and $\displaystyle\sum_{n=1}^{\infty} f_n'$ converge uniformly to functions $f, g : \mathbb{R} \to \mathbb{R}$.

  (b) the functions f, g are continuous.

  (c) $f' = g$.

  (d) it holds that

    (i) $\displaystyle\int_{-1}^{1} f(x) \, dx = 2 \sum_{n=1}^{\infty} \dfrac{1}{\sqrt{n}} \arctan \dfrac{1}{n\sqrt{n}}$ 🛢

    (ii) $\displaystyle\int_{-\pi}^{\pi} x^4 g(x) \, dx = 0$.

(57) Consider the real valued sequence $\{y_n\}_{n \in \mathbb{N}}$ such that forall real valued sequences $\{x_n\}_{n \in \mathbb{N}}$ with $\lim x_n = 0$ the series $\displaystyle\sum_{n=1}^{\infty} x_n y_n$ converges. Prove that the series $\displaystyle\sum_{n=1}^{\infty} |y_n|$ also converges.

🖥 What can you say about the integral $\int_{-\infty}^{\infty} f(t) \, dt$? Does it converge?

58 Let $\{a_n\}_{n\in\mathbb{N}}$ be a decreasing sequence of positive terms. Prove that the series $\sum a_n \sin nx$ converges uniformly throughout $\mathbb{R}$ if and only if $na_n \to 0$.

59 Let $\{a_n\}_{n\in\mathbb{N}}$ be a decreasing sequence of positive terms. Prove that the series $\sum_{n=1}^{\infty} a_n \cos nx$ converges uniformly on $\mathbb{R}$ if and only if the series $\sum_{n=1}^{\infty} a_n$ converges.

60 Let $\mathcal{H}_n$ denote the n-th Harmonic number. Prove the inequality

$$\frac{\pi^2}{6}\left(\zeta(3) - \frac{\pi^2}{12}\right) < \sum_{n=1}^{\infty} \frac{e^{\mathcal{H}_n} \log \mathcal{H}_n}{n^3}$$

🗄

61 Let $f : \mathbb{R} \to \mathbb{R}$ be an integrable function. Prove that

$$S = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} f\left(x - \sqrt{n}\right)$$

converges for almost all x.

62 Prove that the series

$$S = \sum_{n=1}^{\infty} \frac{\cos(\log k)}{k}$$

diverges by first proving that

$$\sum_{n=1}^{N} \frac{\cos \log n}{n} = \sin \log N + \mathfrak{Re}\zeta(1 + i) + \mathcal{O}(N^{-1})$$

63 What is the monotony of the function

$$f(j) = \prod_{i=-j}^{0} \sum_{k=0}^{\infty} \frac{i^k}{k!}, \quad j \in \mathbb{Z}$$

64 Let $f : [-\pi, \pi] \to \mathbb{R}$ be a Riemann integrable function. Prove that

$$\lim_{n\to+\infty} \int_{-\pi}^{\pi} f(x) \cos nx \, dx = \lim_{n\to+\infty} \int_{-\pi}^{\pi} f(x) \sin nx \, dx = 0$$

🗄You might consider ideas from this link.

65 Prove, without using special functions, that the integral

$$\int_0^\pi \frac{\ln x}{x + \pi} \, dx \text{ converges.}$$

66 Let $f_n(x) : [0, 1] \to \mathbb{R}$ be a sequence of functions converging uniformly to a function f. Prove that

$$\lim_{n\to+\infty} \int_{1/n}^1 f_n(x) \, dx = \int_0^1 f(x) \, dx$$

67 Let $f, g : \mathbb{R} \to \mathbb{R}$ be 1 periodic and continuous functions. Prove that

$$\lim_{n\to+\infty} \int_0^1 f(x)g(nx) \, dx = \int_0^1 f(x) \, dx \int_0^1 g(x) \, dx$$

68 Let $f, g : [0, 1] \to \mathbb{R}$ be continuous functions such that $0 < f(x) < cg(x)$ forall $x \in (0, 1)$. for some constant c. Evaluate the limit:

$$\ell = \lim_{n\to+\infty} \int_0^1 \cdots \int_0^1 \frac{f(x_1) + \cdots + f(x_n)}{g(x_1) + \cdots + g(x_n)} \, d(x_1, \ldots, x_n)$$

69 Evaluate the limit

$$\ell = \lim_{n\to+\infty} \int_0^1 \cdots \int_0^1 \frac{x_1^2 + \cdots + x_n^2}{x_1 + \cdots + x_n} \, d(x_1, \ldots, x_n)$$

70 Let $f : (0, +\infty) \to \mathbb{R}$ such that, forall $x > 0$, the limit $\lim_{n\to+\infty} f(nx) \in \mathbb{R}$. Examine if the limit $\lim_{x\to+\infty} f(x)$ exists in $\mathbb{R}$ if:

(a) f is a continuous function.

(b) f is an arbitrary function.

71 (a) Give an example of a bounded function $f : (0, +\infty) \to \mathbb{R}$ such that the limit $\ell = \lim_{x\to0^+} f(x)$ does not exist.

(b) If f is a function such as described in (a) then examine if the following limits exist.

   (i) $\ell_1 = \lim_{x\to0^+} xf(x)$

   (ii) $\ell_2 = \lim_{x\to0^+} (1 - x)f(x)$

72 Find all polynomials P such that $\sin P(x)$ is periodic.

73) Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Prove that

$$\lim_{n \to +\infty} \int_a^b \frac{f(x)}{3 + 2\cos nx}\,dx = \frac{1}{\sqrt{5}} \int_a^b f(x)\,dx$$

74) Evaluate

$$\ell = \lim_{n \to +\infty} n^{-n^2} \left[\prod_{k=0}^{n-1} \left(n + \frac{1}{2^k} \right) \right]^n$$

75) Prove that

$$\min_{a_i \in \mathbb{R}} \int_0^1 \left| x^n + a_1 x^{n-1} + \cdots + a_n \right|\,dx = \frac{1}{4^n}$$

76) Let p, q be two points and γ be a curve passing through these two points. Prove that

(a) $\gamma'(t) \cdot u \leqslant \|\gamma'(t)\|$ where u is an arbitrary unit vector.

(b) that the segment of the curve γ between the points p and q has length at least equal to the distance $\|q - p\|$ by considering as $u = \frac{q-p}{\|q-p\|}$.

🝡

77) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f'(x) = 0$ forall $x \in \mathbb{Q}$. Does it necessarily follow that f is constant throughout $\mathbb{R}$? Explain your answer.

78) Find all functions $f : \mathbb{R} \to \mathbb{R}$ that preserve convergent series. (*That is a function preserves convergent series in the sense mentioned above if* $\sum f(a_n)$ *converges whenever* $\sum a_n$ *converges.*) 🝡

79) Find a function f defined on $\mathbb{R}$ that is not constant and in every interval (x_1, x_2) there exists an a such that

$$f(a) \geqslant \max\{f(x_1),\ f(x_2)\}$$

🝡

🝡The conclusion of this exercise is to show that the line is the shortest distance between two points.

🝡The answer to this difficult question is that the only functions with this property are of the form $f(x) = \lambda x,\ x \in (-\delta, \delta)$.

🝡The function which is equal to 1 everywhere except at 0 on which it is equal to 0 is such a function. The non continuous functions of the Cauchy equation are also such functions and

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \text{ or if } x \text{ is irrational} \\ q & \text{if } x \neq 0 \text{ is rational with } x = p/q \text{ where } q > 0 \text{ and } \gcd(p, q) = 1 \end{cases}$$

80) Examine if there exists a function $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x)) = x^2 + 1 \text{ forall } x \in \mathbb{R}$$

81) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that:

$$f(x) = f(x + 1) = f(x + 2\pi)\quad,\quad \forall x \in \mathbb{R}$$

Prove that f is constant.

82) A function $f : \mathbb{R} \to \mathbb{R}$ is defined as

$$f(x) = \lim_{n \to +\infty} \frac{x^{2n} - 1}{x^{2n} + 1}$$

Where is f continuous?

83) Given a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that forall $x \in \mathbb{R} \setminus \{0, 1\}$

$$\int_0^x f(t)\,dt > \int_x^1 f(t)\,dt \qquad (1)$$

prove that $\int_0^1 f(t)\,dt = 0$.

84) Let $f : [a, b] \to \mathbb{R}$ be a continuously differentiable function such that $f(a) = f(b) = 0$ and $\int_a^b f^2(t)\,dt = 1$. Prove that:

(a) $\int_a^b x f(x) f'(x)\,dx = -\dfrac{1}{2}$

(b) $\int_a^b \left(f'(x) \right)^2\,dx \int_a^b x^2 f^2(x)\,dx > \dfrac{1}{4}$

85) Let

$$f(x) = \sin x \sin(2x) \sin(4x) \cdots \sin(2^n x)$$

Prove that

$$|f(x)| \leqslant \frac{2}{\sqrt{3}} \left| f\left(\frac{\pi}{3} \right) \right|$$

86) Prove that for every $x \in \mathbb{R}$ the inequality

$$\frac{x^{2n}}{(2n)!} + \frac{x^{2n-1}}{(2n-1)!} + \cdots + \frac{x^2}{2!} + x + 1 > 0$$

holds.

(87) Prove that for arbitrary real numbers $a_1, a_2, \ldots, a_n$ the following inequality holds.

$$\sum_{m,n=1}^{k} \frac{a_m a_n}{m+n} \geqslant 0$$

≋

(88) Let $\{a_n\}_{n \in \mathbb{N}}$ be a sequence of positive real numbers. Prove that

$$\limsup_{n \to +\infty} \left(\frac{a_1 + a_{n+1}}{a_n} \right)^n \geqslant e$$

≋

(89) Let $\mathcal{C}$ denote the Cantor set. We define the function $\chi_\mathcal{C} : [0,1] \to \mathbb{R}$ as follows:

$$\chi_\mathcal{C} = \begin{cases} 1 & , \quad x \in \mathcal{C} \\ 0 & , \quad \text{elsewhere} \end{cases}$$

(a) Prove that $\chi_\mathcal{C}$ is Riemann integrable.

(b) Evaluate $\int_0^1 \chi_\mathcal{C}(x)\,dx$.

(90) Prove that the function $f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ defined as

$$f(\mathbf{x}) = \frac{\mathbf{x}}{\|\mathbf{x}\|^a} \ , \quad a > 0$$

is a vector field but its domain is not star-shaped.

(91) Does the ordered field of the rational functions satisfy the axiom of completeness? Explain your answer.

≋A solution goes along these lines:

$$\sum_{m,n=1}^{k} \frac{a_m a_n}{m+n} = \sum_{m,n=1}^{k} \int_0^1 a_m a_n t^{m+n-1}\,dt$$

$$= \int_0^1 \left(\sum_{m,n=1}^{k} a_m a_n t^{m+n-1} \right) dt$$

$$= \int_0^1 \left(\sum_{m=1}^{k} a_m t^{m-1/2} \right)^2 dt$$

$$\geqslant 0$$

In fact the above inequality tells us that the matrix $\left[\dfrac{1}{m+n} \right]_{m,n=1}^{k}$ is positive semidefinite.

≋This is a very difficult exercise. One solution may be found at M. Hata's notes. Another solution is to contradict the result and move along those lines.

(92) Let $f : [2, +\infty) \to \mathbb{R}$ be a uniformly continuous function. Prove that the integral

$$\mathcal{I} = \int_2^\infty \frac{f(x)}{x^2 \log^2 x}\,dx$$

converges.

(93) Let $f : [0, +\infty) \to \mathbb{R}$ be a continuous and strictly convex function such that $\displaystyle \lim_{x \to +\infty} \frac{f(x)}{x} = +\infty$. Prove that the integral $\int_0^\infty \sin f(x)\,dx$ converges but not absolutely.

≋

(94) Examine if there exists a continuous function $f : [1, +\infty) \to \mathbb{R}$ such that $f(x) > 0$ forall $x \in [1, +\infty)$ such that $\int_1^\infty f(x)\,dx$ converges whereas $\int_1^\infty f^2(x)\,dx$ diverges.

(95) Let $f : [a,b] \to \mathbb{R}$ be a Riemann integrable function. If $f(x) = 0$ forall rationals of the interval $[a,b]$ then prove that $\int_a^b f(x)\,dx = 0$.

(96) Prove that there exists no rational function such that

$$f(n) = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

forall $n \in \mathbb{N}$.

(97) Let $f : \mathbb{R} \to (0, +\infty)$ be a function such that forall $x \in \mathbb{R}$ it holds that

$$f(x) \log f(x) = e^x \tag{1}$$

Evaluate the limit

$$\ell = \lim_{x \to +\infty} \left(1 + \frac{\log x}{f(x)} \right)^{f(x)/x}$$

(Romania , 1986)

(98) Let $n \in \mathbb{N}$ and let $f : [-1, 1] \to \mathbb{R}$ be a continuous function such that

$$\int_{-1}^1 x^{2n} f(x)\,dx = 0$$

Prove that f is odd.

≋I currently have no solution to this , demanding , exercise. It was an exam's question.

99 Let $\mathcal{G}$ denote the Catalan constant. Prove that

$$\log\left(1 + \sqrt{2}\right) < \int_0^1 \frac{\tanh x}{x}\, dx < \mathcal{G}$$

100 Evaluate the limit

$$\Omega = \lim_{n\to+\infty} \sum_{k=1}^n \frac{\frac{1}{n}\arctan\left(\frac{k}{n}\right)}{1 + 2\sqrt{1 + \frac{1}{n}\arctan\left(\frac{k}{n}\right)}}$$

(Dan Sitaru)

101 Evaluate the limit

$$\Omega = \lim_{n\to+\infty} \sum_{k=1}^n \arcsin\frac{1}{\sqrt{n^2 + k}}$$

102 Let φ denote Euler's totient function. Evaluate the limit

$$\ell = \lim_{n\to+\infty} \frac{1}{n^2}\sum_{k=1}^n \sin\left(\frac{\pi k}{n}\right)\varphi(k)$$

103 Let $\alpha > 0$. Prove that:

$$\lim_{n\to+\infty} \frac{1}{\log n}\sum_{1\leqslant k\leqslant n^\alpha} \frac{1}{k}\left(1 - \frac{1}{n}\right)^k = \min\{1, \alpha\}$$

104 Let us denote with ζ the Riemann zeta function with $\zeta(0) = -\frac{1}{2}$. Let us also denote with $\zeta^{(n)}$ the n – th derivative of zeta. Evaluate the limit

$$\ell = \lim_{n\to+\infty} \frac{\zeta^{(n)}(0)}{n!}$$

105 Let ζ denote the Riemann zeta function. Evaluate the limit

$$\ell = \lim_{n\to+\infty} n\left(\zeta(2) - \sum_{k=1}^n \frac{1}{k^2}\right)$$

106 Let ζ denote the Riemann zeta function. Evaluate the limits:

The above limit tells us that $\zeta^{(n)}(0) \sim -n!$.

(a) $\ell_1 = \lim_{n\to+\infty} \zeta(n)$

(b) $\ell_2 = \lim_{n\to+\infty} \sqrt[n]{\zeta(n) - 1}$

107 Let ζ denote the Riemann zeta function. Evaluate the limit

$$\ell = \lim_{n\to+\infty} \frac{1}{\zeta(n)}\sum_{k=1}^n \frac{1}{k^n}$$

108 Let $\mathcal{H}_n$ denote the n - th harmonic number. Prove that

$$\lim_{n\to+\infty} n\left(\mathcal{H}_n - \log n - \gamma\right) = \frac{1}{2}$$

109 Let $\widetilde{\mathcal{H}_n} = \sum_{j=1}^n \frac{(-1)^{j-1}}{j}$. Prove that the limit

$$\ell = \lim_{n\to+\infty} n\left[\widetilde{\mathcal{H}_n} - \mathcal{H}_{2n} + \mathcal{H}_n\right]$$

does not exist.

110 Let $\mathcal{H}_n$ denote the n - th harmonic number. Prove that

$$\lim_{n\to+\infty} \left(\mathcal{H}_n - \frac{1}{2^n}\sum_{k=1}^n \binom{n}{k}\mathcal{H}_k\right) = \log 2$$

111 Evaluate the limit

$$\ell = \lim_{n\to+\infty} \frac{(n!)^2}{(1 + 1^2)(1 + 2^2)\cdots(1 + n^2)}$$

112 Let Γ denote the Euler's Gamma function. Prove that

$$\frac{\Gamma\left(\frac{1}{10}\right)}{\Gamma\left(\frac{2}{15}\right)\Gamma\left(\frac{7}{15}\right)} = \frac{\sqrt{5} + 1}{3^{1/10}2^{6/5}\sqrt{\pi}}$$

113 Let $f : [0, +\infty) \to \mathbb{R}$ be an integrable and uniformly continuous function. Prove that $\lim_{x\to+\infty} f(x) = 0$. Does this result hold if we drop the assumption of the *uniformly continuous* ? Explain your answer.

114 Define a function $f : \mathbb{R} \to \mathbb{R}$ such that for every $q \in \mathbb{Q}$ must hold $f(q) \in \mathbb{Q}$ but $f'(q) \notin \mathbb{Q}$.

In fact prove that $\ell = 1$.

115 Let $f : [0, 1] \to \mathbb{R}$ be defined as:

$$f(x) = \begin{cases} 0 & , \quad x \in [0, 1] \cap (\mathbb{R} \setminus \mathbb{Q}) \\ x_n & , \quad x = q_n \in [0, 1] \cap \mathbb{Q} \end{cases}$$

where x_n is a sequence such that $\lim x_n = 0$ and $0 \leqslant x_n \leqslant 1$ and q_n be an enumeration of the rationals of the interval $[0, 1]$. Prove that f is Riemann integrable and that $\int_0^1 f(x)\,dx = 0$.

116 Let f be holomorphic on the open unit disk $\mathbb{D}$ and suppose that

$$\iint\limits_{\mathbb{D}} |f(z)|^2 \; d(x, y) < +\infty$$

If the Taylor expansion of f is of the form $\sum\limits_{n=0}^{\infty} a_n z^n$ then prove that the series $\sum\limits_{n=0}^{\infty} \frac{|a_n|^2}{n+1}$ converges.

117 Let f_n be a sequence of real valued $\mathcal{C}^1$ functions on $[0, 1]$ such that forall $n \in \mathbb{N}$ the following hold:

- ■ $|f_n'(x)| \leqslant \dfrac{1}{\sqrt{x}} \quad (0 < x \leqslant 1)$

- ■ $\int_0^1 f_n(x)\,dx = 0$

Prove that f_n has a convergent subsequence that converges uniformly on $[0, 1]$.

118 Let $\chi_{\mathbb{Q}}$ denote the characteristic function of the rationals in $[0, 1]$. Does there exist a sequence of continuous functions $f_n : [0, 1] \to \mathbb{R}$ such that f_n converges to $\chi_{\mathbb{Q}}$ pointwise?

119 Let $f : [0, 1] \to \mathbb{R}$ be a continuous function such that

$$\int_0^1 f(t)\,dt = \int_0^1 tf(t)\,dt = 1 \tag{1}$$

Prove that $\int_0^1 f^2(t)\,dt \geqslant 4$.

120 Let $f : [0, 1] \to \mathbb{R}$ be a continuous function such that

$$\int_0^1 f(t)\,dt = \int_0^1 tf(t)\,dt \tag{1}$$

Prove that there exists a $c \in (0, 1)$ such that

$$\int_0^c f(t)\,dt = \frac{c}{2} \int_0^c f(t)\,dt$$

121 Let $f : [0, 1] \to \mathbb{R}$ be a continuous function such that

$$\int_0^1 f(t)\,dt = \int_0^1 tf(t)\,dt \tag{1}$$

Prove that there exists a $c \in (0, 1)$ such that

$$cf(c) = 2 \int_c^0 f(t)\,dt$$

122 Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that

$$f'(x) = f^2(x)f(-x) \tag{1}$$

Find an explicit formula for f.

123 Let $f : [0, 1] \to \mathbb{R}$ be a continous function such that $\int_0^1 f(t)\,dt = 1$ and

$$\int_0^1 (1 - f(x))\,e^{-f(x)}\,dx \leqslant 0 \tag{1}$$

Prove that $f(x) = 1$ forall $x \in \mathbb{R}$.

124 Let $f : [a, b] \to [0, +\infty)$ be a continous and not everywhere 0 function. Prove that

$$\lim_{n \to +\infty} \frac{\int_a^b f^{n+1}(t)\,dt}{\int_a^b f^n(t)\,dt} = \sup_{x \in [a,b]} f(x)$$

125 Examine if there exists a continuous function $f : [1, +\infty) \to \mathbb{R}$ such that $f(x) > 0$ forall $x \in [1, +\infty)$ and $\int_1^\infty f(t)\,dt$ converges whereas $\int_1^\infty f^2(t)\,dt$ diverges. 🗄

126 Let $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and consider the function

$$f(x) = a_1 \tan x + a_2 \tan \frac{x}{2} + \cdots + a_n \tan \frac{x}{n}$$

where $a_1, a_2, \ldots, a_n \in \mathbb{R}$ and $n \in \mathbb{N}$. If $|f(x)| \leqslant |\tan x|$ for all $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ then prove that

$$\left| a_1 + \frac{a_2}{2} + \cdots + \frac{a_n}{n} \right| \leqslant 1$$

🗄Do the same exercise with the extra assumption that f is uniformly continuous.

(127) Let $f : [0, 1] \to \mathbb{R}$ be a twice differentiable function with a continuous second derivative. If n is a natural number greater than 1 such that

$$\sum_{k=1}^{n-1} f\left(\frac{k}{n}\right) = -\frac{f(0) + f(1)}{2}$$

then prove that

$$\left(\int_0^1 f(t)\, dt\right)^2 \leqslant \frac{1}{5! n^4} \int_0^1 \left(f''(t)\right)^2 dt$$

(128) Prove that every function $f : \mathbb{Q} \to \mathbb{Q}$ can be written as the sum of two $1 - 1$ functions $g, h : \mathbb{Q} \to \mathbb{Q}$.

(129) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that any rational number is its period but any irrational is not. Also, prove that there exists no function $g : \mathbb{R} \to \mathbb{R}$ such that any irrational is its period and any rational is not.

(130) Prove that the function

$$f(x) = \begin{cases} \sin\left(\ln^2 x\right) & , \quad x > 0 \\ 0 & , \quad x = 0 \end{cases}$$

has a primite on $[0, +\infty)$.

(Constanza , 2009)

(131) Let $\mathbb{F}$ be an ordered field. Define $f : \mathbb{F} \to \mathbb{F}$ such that it satisfies

$$|f(x) - f(y)| \leqslant |x - y|^2, \quad \forall x, y \in \mathbb{F}$$

Is $\mathbb{F}$ necessarily Archimidean?

(132) Compute the limit:

$$\ell = \lim_{n \to +\infty} \frac{1}{n^2} \sum_{1 \leqslant i \leqslant j \leqslant n} \ln\left(\frac{3n - i}{3n + i}\right) \ln\left(\frac{3n - j}{3n + j}\right)$$

(133) Compute the limit

$$\ell = \lim_{n \to +\infty} \frac{1}{n} \sum_{k=1}^n \frac{k + n}{n + 2\sqrt{n^2 + n + k}}$$

(134) Evaluate the sum

$$S_n = \sum_{k=1}^n \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} \frac{2^k}{(i_1 + 1)(i_2 + 1) \cdots (i_k + 1)}$$

(135) Compute the limit

$$\ell = \lim_{n \to +\infty} \left[\sum_{i=1}^n \sum_{j=1}^n \frac{1}{i^2 + j^2} - \frac{\pi \log n}{2}\right]$$

(136) Let ζ denote the Riemann zeta function. Prove that

$$\lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^T \frac{\zeta(\frac{3}{2} + it)}{\zeta(\frac{3}{2} - it)}\, dt = \frac{1}{\zeta(3)}$$

(137) Let $\lfloor \cdot \rfloor$ denote the floor function. Prove that forall $n \in \mathbb{N}$ it holds that

$$\left\lfloor \left(\sum_{k=n}^\infty \frac{1}{k^3}\right)^{-1} \right\rfloor = 2n(n - 1)$$

(138) Let Γ denote the Euler's Gamma function. Prove that

$$x^{1-s} < \frac{\Gamma(x + 1)}{\Gamma(x + s)} < (x + 1)^{1-s}, \quad x > 0, \ 0 < s < 1$$

(139) (a) Let $a > 0$. Evaluate the integral

$$\mathcal{J}(a) = \int_0^a \log\left(1 + \tan a \tan x\right) dx$$

(b) Evaluate the limit $\lim_{a \to 0} \dfrac{\mathcal{J}(a)}{a^3}$.

(140) Let $\max\{, \cdot,\}$ denote the max function. What can you say about the integrals?

$$\mathcal{J}_1 = \int_0^1 \int_0^1 \frac{x - y}{\max\{x^3, y^3\}}\, d(x, y)$$

This inequality is better known as Gautchi's Inequality.
You might as well evaluate the integral first by making the substitution $y = a - x$.

$$\mathcal{J}_2 = \int_0^1 \int_0^1 \frac{x - y}{\max\{x^3, y^3\}} \, d(y, x)$$

What does this exercise teach you? 🍩

(141) Let $\langle \cdot, \cdot \rangle$ denote the usual inner product of $\mathbb{R}^m$. Evaluate the integral

$$\mathcal{M} = \int_{\mathbb{R}^m} \exp\left(-(\langle x, \mathcal{S}^{-1} x \rangle)^a\right) \, dx$$

where $\mathcal{S}$ is a positive symmetric $m \times m$ matrix and $a > 0$.

(142) Prove that for an entire function f holding $\lim_{z \to \infty} \frac{f(z)}{z} = 0$ then f is constant.

(143) Let $f : \mathbb{C} \to \mathbb{C}$ be an analytic and $1 - 1$ function and let $\mathbb{D}$ be the open unit disk. Prove that

$$\iint_{\mathbb{D}} |f'(z)| \, dz = \text{area}(f(\mathbb{D}))$$

🍩

(144) Let $n \in \mathbb{N}$ and f be an entire function. Prove that for any arbitrary positive numbers a, b it holds that

$$\frac{\int_0^{2\pi} e^{-int} f(z + ae^{it}) \, dt}{\int_0^{2\pi} e^{-int} f(z + be^{it}) \, dt} = \left(\frac{a}{b}\right)^n$$

(145) Let $a, b \in \mathbb{C}$ such that $|b| < 1$. Prove that

$$\frac{1}{2\pi} \oint_{|z|=1} \left|\frac{z - a}{z - b}\right|^2 |dz| = \frac{|a - b|^2}{1 - |b|} + 1$$

(146) Define

$$f(z) = \frac{1}{z} \cdot \frac{1 - 2z}{z - 2} \cdots \frac{1 - 10z}{z - 10}$$

Evaluate the contour integral $\oint_{|z|=100} f(z) \, dz$.

(147) Prove that there does not exist a sequence $\{p_n(z)\}_{n \in \mathbb{N}}$ of complex polynomials such that $p_n(z) \to \frac{1}{z}$ uniformly on $\mathcal{C}_R = \{z \in \mathbb{C} \mid |z| = R\}$.

(148) Let f be a meromorphic function on a (connected) Riemann Surface X. Show that the zeros and the poles of f are isolated points.

(149) Let us prove that $0 = 1$. We begin by stating Picard's Little Theorem:

> **Theorem**
>
> If a function $f : \mathbb{C} \to \mathbb{C}$ is entire and non-constant, then the set of values that $f(z)$ assumes is either the whole complex plane or the plane minus a single point.

Let us now consider $g(z) = e^z$ which is definitely complex differentiable. Since the composition of complex differentiable functions is also complex differentiable then the function

$$f(z) = g(g(x)) = e^{e^z}$$

is also complex differentiable. Also, f is not constant; that is for sure. Since there exists no z such that $e^z = 0$ then 0 and 1 are not in the range of f. However, this is an obscurity unless $0 = 1$.

Find the flaw in the above argument. 🍩

(150) Let $\psi^{(n)}$ denote the n - th polygamma function and let $n \in \mathbb{N} \cup \{0\}$. Prove that

$$\frac{\psi^{(n)}(z)}{\psi^{(n+1)}(z)} \geqslant \frac{\psi^{(n+1)}(z)}{\psi^{(n+2)}(z)} \quad, \quad z > 0$$

🍩

(151) Consider the points $O(0, 0)$ and $A(1, 0)$. Let $\Gamma(x, y)$ be a point of the plane such that $y > 0$. Set $\varphi(x, y)$ to be the angle that is defined by $O\Gamma$ and $A\Gamma$. (the one that is less than π.) Prove that the function $\varphi(x, y)$ is harmonic.

🍩 Does symmetry help you to evaluate the integral? Where is the flaw in this method? Give a brief explanation.

🍩 This is known as Lusin Area Integral Formula.

🍩 The flaw is not in the theorem!

🍩 Actually the above inequality is a consequence of a stronger one namely this:

$$\psi^{(m)}(z)\psi^{(n)}(z) \geqslant \psi^{\left(\frac{m+n}{2}\right)}(z)$$

whenever $\frac{m+n}{2} \in \mathbb{N}$. The proof of it may be found at Joy of Mathematics.

Multivariable Calculus

152 Let f be analytic in the unit disk $\mathbb{D}$. Suppose that $\mathrm{Re}(f(z)) \geqslant 0$ forall $z \in \mathbb{D}$ and that $f(0) = 1$. Prove that

$$\frac{1 - |z|}{1 + |z|} \leqslant \mathrm{Re}\left(f(z)\right) \leqslant |f(z)| \leqslant \frac{1 + |z|}{1 - |z|}$$

153 Let $f(z) \in \mathbb{Q}[z]$ be irreducible with degree $n > 1$. If f has a root on the unit circle then n is even and

$$z^n f\left(\frac{1}{z}\right) = f(z)$$

154 Find all smooth functions g with domain $\mathbb{R}^2 \setminus \{(0,0)\}$ such that

$$\nabla g = \left(-\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2}\right)$$

🛢

155 Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function. Prove that there exist functions g_i, $i = 1, \ldots, n$ such that

$$f(x_1, x_2, \ldots, x_n) - f(0, 0, \ldots, 0) = \sum_{i=1}^{n} x_i g_i(x_1, x_2, \ldots, x_n)$$

156 Given the curve $\gamma(t) = e^{-t}(\cos t, \sin t)$, $t \geqslant 0$

  (a) Sketch its graph.

  (b) Evaluate the length of the curve as well as the following line integrals

    (i) $\oint_\gamma (x^2 + y^2)\, ds$ (ii) $\oint_\gamma (-y, x) \cdot d(x, y)$

(Question from a Real Analysis Exam
University of Ioannina , Greece)

157 (a) Let $\mathbb{D} \subset \mathbb{R}^2$ be the unit disk and $\partial\mathbb{D}$ be its positive oriented boundary. Evaluate the following line integral

$$\oint_{\partial\mathbb{D}} \left(x - y^3, x^3 - y^2\right) \cdot d(x, y)$$

🛢Such functions do not exist. Reason being that if we consider $\mathcal{C}(R)$ to be a circle of centre 0 and radius R then $0 = \oint_{\mathcal{C}(R)} \nabla g\, dr > 0$ which is obsiously an obscurity.

(b) Can you deduce if the function

$$\bar{f}(x, y) = \left(x - y^3, x^3 - y^2\right)$$

is a vector field by basing your reasoning **solely** on question **(a)** ?

(Question from a Real Analysis Exam
University of Ioannina , Greece)

158 (a) Let $f \in \mathcal{C}^2(\mathbb{R})$ such that $\mathrm{div}\,\mathrm{grad}(f) = 0$ and $\mathbb{D} \subseteq \mathbb{R}^2$ be a $\mathcal{C}^1$ normal set. Prove that

$$\oint_{\partial\mathbb{D}} \left(\frac{\partial f}{\partial y}, -\frac{\partial f}{\partial x}\right) \cdot d(x, y) = 0$$

(b) Examine if

$$\bar{f}(x, y) = (2x \cos y, -x^2 \sin y)$$

is a conservative field and if so, find a scalar potential.

(Question from a Real Analysis Exam
University of Ioannina , Greece)

159 Prove that for every $c > 0$ the set

$$\mathcal{B}_{f,g} = \{(x, y, z) \in \mathbb{R}^3 : (x - f(z))^2 + (y - g(z))^2 \leqslant c \ , \ z \in [a, b]\}$$

has the same volume for every function $f, g : [a, b] \to \mathbb{R}$.

160 Consider the subset of $\mathbb{R}^3$

$$\mathcal{B} = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leqslant z \leqslant a\}, \ a > 0$$

(a) Evaluate

  (i) the volume of $\mathcal{B}$.

  (ii) the triple integral

$$\mathcal{T} = \iiint_{\mathcal{B}} (x^2 + y^2)z\, d(x, y, z)$$

  (iii) the area of the boundary of $\mathcal{B}$.

  (iv) the surface integral

$$\mathcal{S} = \oint_{\partial\mathcal{B}} \sqrt{1 + 4z^2}\, d\sigma$$

(b) Express the volume of $\mathcal{B}$ through a suitable continuously differentiable $G : \mathbb{R}^3 \to \mathbb{R}^3$ and through a suitable surface integral.

161 Prove that the work

$$W = -\oint_\gamma \frac{(x, y, z)}{(x^2 + y^2 + z^2)^{3/2}} \cdot d(x, y, z)$$

produced along a $\mathcal{C}^1$ oriented curve γ of $\mathbb{R}^3 \setminus \{(0, 0, 0)\}$ depends only on the distances of starting and ending point of γ about the origin.

162 Let $\mathcal{V}_n(R)$ be the volume of the ball of center 0 and radius $R > 0$ in $\mathbb{R}^n$. Prove that for $n \geqslant 3$ it holds that

$$\mathcal{V}_n(1) = \frac{2\pi}{n}\mathcal{V}_{n-2}(1)$$

163 Let $\mathbb{S}$ denote the area bounded by the curves $x^2 y = 1$ and $x^2 y = 2$ as well as the lines $y = x$ and $y = 2x$ and let γ denote its negative oriented boundary. Evaluate

$$\mathcal{J} = \oint_\gamma \left(e^{-x^2} - 6y\right) dx + \left(4x - 7y^7\right) dy$$

164 Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a continously differentiable function and let $\mathcal{C}_r$ be the circle of origin $(0, 0)$ and radius $r > 0$. Prove that:

$$\frac{1}{2\pi} \lim_{r \to 0} \frac{1}{r} \oint_{\mathcal{C}_r} u \, ds = u(0, 0)$$

165 Let $f(\mathbf{x}) = \mathbf{x}^\top Q \mathbf{x}$ where $\mathbf{x}^\top = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and Q is the diagonal matrix

$$Q = \begin{pmatrix} q_1 & 0 & \cdots & 0 \\ 0 & q_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & q_n \end{pmatrix} \quad q_i \in \mathbb{R}, \ i = 1, \ldots, n$$

(a) Give the derivative as well as the Hessian matrix of f.

(b) Give conditions for the q_i such that f has **a)** a local maximum **b)** a local minumum and **c)** neither of the previous ones.

(c) Compute the Taylor polynomial of degree k of f around $\mathbf{x} = \mathbf{0}$ for all $k \in \mathbb{N}$.

166 Let $\mathbb{S} = [0, 1] \times [0, 1] \subset \mathbb{R}^2$. Evaluate the integral

$$\mathcal{J} = \iint_{\mathbb{S}} \max\{x, y\} \, d(x, y)$$

Hint: It holds that

$$\max\{x, y\} = \begin{cases} x & , \quad 0 \leqslant y \leqslant x \leqslant 1 \\ y & , \quad 0 \leqslant x \leqslant y \leqslant 1 \end{cases}$$

Hence

$$\int_0^1 \int_0^1 \max\{x, y\} \, d(x, y) = \int_0^1 \int_0^x x \, d(y, x) +$$
$$+ \int_0^1 \int_0^y y \, d(x, y)$$
$$= 2 \int_0^1 \int_0^x x \, d(y, x)$$
$$= 2 \int_0^1 x^2 \, dx$$
$$= \frac{2}{3}$$

🝥

167 Let M be the interesection of the elliptic cylinder

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \leqslant 1 \text{ and of the ellipsoid}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leqslant 1 \quad a > 0, \ b > 0, \ c > 0$$

For all $n \in \mathbb{N}$ evaluate the integrals

$$I_n = \iiint_M \left(a^2 b^2 - b^2 x^2 - a^2 y^2\right)^{n - \frac{1}{2}} d(x, y, z)$$

(Question from a Real Analysis Exam University of Ioannina, Greece)

168 Let $\mathcal{C} = [0, 1] \times [0, 1] \times \cdots \times [0, 1] \subseteq \mathbb{R}^n$ be the unit cube. Define the function

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🝥 An interpretation of this integral; if you have two independent uniform $(0, 1)$ random variables, the expected value of the maximum is $\frac{2}{3}$. (And the expected value of the minimum is $\frac{1}{3}$.) More generally: if you have n independent uniform $(0, 1)$ random variables, the expected value of the maximum is $\frac{n}{n+1}$. In more detail: if you order these random variables after the fact so that $Y_1 \leqslant Y_2 \leqslant \cdots \leqslant Y_n$, then the expected value of Y_k is $\frac{k}{n+1}$. (The general name for this sort of reasoning is order statistics.)

General Topology

$$f(x_1, x_2, \ldots, x_n) = \frac{x_1 x_2 \cdots x_n}{x_1^{a_1} + x_2^{a_2} + \cdots + x_n^{a_n}}$$

where a_i arbitrary positive constants. For which values of $a_i > 0$ is the value of the integral $\int_e f$ finite?

(169) Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x) \geqslant 0$ and $\int_{-\infty}^{\infty} f(t) \, dt = 1$. For $r \geqslant 0$ we define

$$I_n(r) = \int \cdots \int_{x_1^2 + x_2^2 + \cdots + x_n^2 \leqslant r} f(x_1) f(x_2) \cdots f(x_n) \, d(x_1, x_2, \ldots, x_n)$$

Evaluate $\lim I_n(r)$.

(170) Let $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$. Let $\mathbb{A}$ denote the area measure on $\mathbb{D}$ normalised so that $\mathbb{A}(\mathbb{D}) = \pi$. Verify or disprove that

$$\iint_{\mathbb{D}} \left| \log\left(\frac{e}{1-z}\right) \right|^2 \, d\mathbb{A} = \frac{\pi^3}{6}$$

(171) For a given function $f : \mathbb{R}^3 \to \mathbb{R}^3$ the $\int_{\mathbb{R}^3} |f(x)| \, dx$ exists. If for every plane $\mathcal{P}$ of $\mathbb{R}^3$ it holds that $\int_{\mathcal{P}} f(x) \, ds = 0$ then prove that f is the zero function. 🍥

(172) Let $A \subseteq \mathbb{R}^n$. If A is Jordan measurable and has zero measure prove that

$$\int_A 1 \, d\bar{x} = 0$$

(173) Find a countable and dense subset of $\mathbb{R} \setminus \mathbb{Q}$ with respect to the usual topology.

(174) Let $\mathcal{X} = [0, +\infty) \cup \{+\infty\}$. We endow it with the metric

$$\rho(x, y) = |\arctan x - \arctan y|$$

Prove that under this metric $\mathcal{X}$ is separable, complete and compact.

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(175) Does there exist an enumeration $\{q_n \in \mathbb{Q} : n \in \mathbb{N}\}$ of $\mathbb{Q}$ such that

$$\mathbb{R} \neq \bigcup_{n=1}^{\infty} \left(q_n - \frac{1}{n}, q_n + \frac{1}{n} \right)$$

(176) Prove that there does not exist an $1 - 1$ and continuous mapping from $\mathbb{R}^2$ to $\mathbb{R}$.

(177) Let Ω be a metric space. Suppose that every bounded subset of Ω has at least one accumulation point. Prove that Ω is a complete metric space.

(178)   (a) Let (X, ρ) be a compact metric space and let $f : X \to X$ be an isometry. Prove that f is onto.

   (b) Prove that the ℓ^2 space (that is the space of the sequences for which $\sum_{n=1}^{\infty} x_n^2$ converges) is not compact endowed by the metric

$$\rho(x_n, y_n) = \sqrt{\sum_{n=1}^{\infty} (x_n - y_n)^2}$$

(179) Prove that there exists no continuous and $1 - 1$ map (depiction) from a sphere to a proper subset of it.

(180) Is the set $\mathcal{S} = \mathbb{R}^2 \setminus \mathbb{Q} \times \mathbb{Q}$ complete? Give a brief explanation.

(181) Let $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$. Endow it with the metric

$$d(x, y) = \left| \frac{1}{x} - \frac{1}{y} \right|$$

   (a) Show that the sequence $a_n = n$ is a Cauchy one.

   (b) Is the sequence $\frac{1}{n}$ a Cauchy one?

   (c) Show that any sequence a_n in $\mathbb{R}^+$ converges in $\mathbb{R}^+$ in the metric d above if and only if it converges in $\mathbb{R}$ in the standard metric $|x - y|$ and that the limits in the two cases are equal.

(182) Let us define the following function:

$$f(x) = \begin{cases} x & , \quad 0 \leqslant x < 1 \\ 1 & , \quad x > 1 \end{cases}$$

as well as $d_m(x, y) = f(|x - y|)$.

(a) Show that d_m is a metric on $\mathbb{R}$. *You may call it the mole metric. If points are close (closer than one meter), their distance is the usual one, but are they far apart (more than one meter) we do not distinguish between their distances; they are just far apart.*

(b) Show that $\mathbb{R}$ endowed with the above metric is complete and bounded but not compact. Is it totally bounded? Why / Why not?

183) Prove that the set $\mathbb{R}^2 \setminus \{0, 0\}$ is not simply connected. ▤

184) Find a sequence of open sets $\{G_n\}_{n \in \mathbb{N}}$ of $\mathbb{R}$ such that

$$\mathbb{Z} = \bigcap_{n=1}^{\infty} G_n$$

▤

185) (a) Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$. Prove that the set

$$\mathcal{D}(\theta) = \{(\cos 2n\pi\theta, \sin 2n\pi\theta) \in \mathbb{R}^2 : n \in \mathbb{N}\}$$

is a dense subset of the circle $\mathbb{S}^1 : x^2 + y^2 = 1$.

(b) Find a countable and dense subset of $\mathbb{R} \setminus \mathbb{Q}$ with respect to the usual metric.

186) Let us denote $\mathbb{S}^2$ the unit sphere that is the set

$$\mathbb{S}^2 = \left\{(x, y, z) \in \mathbb{R}^3 \,\middle|\, x^2 + y^2 + z^2 = 1\right\}$$

If $f : \mathbb{S}^2 \to \mathbb{S}^2$ is a continuous function such that $f(x) \neq f(-x)$ forall $x \in \mathbb{S}^2$ then prove that f is onto.

187) Examine if there exist non constant functions $f : \mathbb{R} \to \mathbb{R}$ that map any open interval onto a closed one.

188) Let (X, d) be a complete and a compact metric space. Prove that there exists a unique number $r = r(X, d)$ with the property:

For all $n \in \mathbb{N}$ and for all $x_i, i = 1, 2, \ldots, n$ there exists $z \in X$ such that

$$\frac{1}{n} \sum_{i=1}^{n} d(z, x_i) = r$$

189) Prove that a metric space (X, d) containing infinite points, where d is the discrete metric, is not compact.

190) Prove that the set

$$A = \left\{(x, y) \in \mathbb{R}^2 \mid y \cos x + x \sin y = 1\right\}$$

is not path-connected with respect to the relative topology of $\mathbb{R}^2$.

191) Find (or construct) a continuous function from the positive rationals that is onto the real numbers.

192) Prove that double inequality

$$\max_{1 \leqslant j \leqslant p} \sqrt{\sum_{i=1}^{q} a_{ij}^2} \leqslant \left\| \begin{pmatrix} a_{11} & \cdots & a_{1p} \\ \vdots & \ddots & \vdots \\ a_{q1} & \cdots & a_{qp} \end{pmatrix} \right\|_2 \leqslant \sqrt{\sum_{i=1}^{q} \sum_{j=1}^{p} a_{ij}^2}$$

▤Well , the problem actually is not of an analysis nature but that of Algebraic Topology. Try to construct a deformation retraction from $\mathbb{R}^2 \setminus \{0, 0\}$ to $\mathbb{S}^1$ (the unit circle). For example take $f(x) = \dfrac{x}{\|x\|}$. Then the fundamental groups are isomorphic , however $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$ and hence the fundamental group is not trivial. Therefore , the set is not simply connected.

▤Simply take

$$G_n = \bigcup_{m \in \mathbb{Z}} \left(m - \frac{1}{n}, m + \frac{1}{n}\right)$$

Integrals and Series

193 Evaluate

$$\mathcal{I} = -\int_1^\infty \sum_{n=0}^\infty \frac{dx}{(n+x)^3}$$

194 Let $a \geqslant -1$. Evaluate

$$\mathcal{I} = \int_0^{\pi/2} \log\left(1 + a\sin^2 x\right) dx$$

195 Let $n \in \mathbb{N} \mid n > 2$. Prove that

$$\int_0^\infty \frac{\log\left(\frac{1}{x}\right)}{(1+x)^n} dx = \frac{1}{n-1} \sum_{k=1}^{n-2} \frac{1}{k}$$

196 Evaluate the integral

$$\mathcal{I} = \int_0^1 \frac{\arctan \dfrac{x}{x+1}}{\arctan \dfrac{1+2x-2x^2}{2}} dx$$

(Russian Mathematical Olympiad)

197 For any positive integer n, let $\langle n \rangle$ denote the closest integer to $\sqrt{n}$. Evaluate the sum

$$\mathcal{S} = \sum_{n=1}^\infty \frac{2^{\langle n \rangle} + 2^{-\langle n \rangle}}{2^n}$$

(Putnam 2001)

198 Prove that

$$\int_0^1 \prod_{n=1}^\infty (1-x^n)\, dx = \frac{4\pi\sqrt{3}\sinh\frac{\pi\sqrt{23}}{3}}{\sqrt{23}\cosh\frac{\pi\sqrt{23}}{2}}$$

199 Evaluate the integral

$$\mathcal{I} = \int_0^1 \frac{\arctan\sqrt{2+x^2}}{(1+x^2)\sqrt{2+x^2}} dx$$

🗇

🗇 This integral is known with the name "Ahmed's integral".

200 Let $a \in \mathbb{R}$. Evaluate the integral

$$\mathcal{I} = \int_{-\infty}^\infty \frac{\cos ax}{e^x + e^{-x}} dx$$

🗇

201 Evaluate the integral

$$\mathcal{I} = \int_0^\infty \frac{x^2 - 4}{x^2 + 4} \frac{\sin 2x}{x} dx$$

202 Evaluate the double series

$$\mathcal{S} = \sum_{k=1}^\infty \frac{(-1)^{k-1}}{k} \sum_{n=0}^\infty \frac{1}{k2^n + 1}$$

(Putnam 2016)

203 Evaluate the integral

$$\mathcal{I} = \int_0^1 \left(\frac{1}{1-x} + \frac{1}{\ln x}\right) dx$$

🗇

204 Let $\psi^{(1)}$ denote the trigamma function. Evaluate the sum

$$\mathcal{S} = \sum_{n=1}^\infty (-1)^{n-1}\left(\psi^{(1)}(n)\right)^2$$

(Cornel Ioan Valean)

205 Let Li_2 denote the dilogarithm function and Γ denote the Gamma function. Prove that

$$\int_0^1 \left(\mathrm{Li}_2\left(e^{-2\pi ix}\right) + \mathrm{Li}_2\left(e^{2\pi ix}\right)\right)\log\Gamma(x)\, dx = \frac{\zeta(3)}{2}$$

where ζ is the Riemann zeta function.

🗇 The evaluation of this integral allows to tell that

$$\mathfrak{Re}\left[\psi^{(0)}\left(\frac{3}{4} - \frac{ia}{4}\right) - \psi^{(0)}\left(\frac{1}{4} - \frac{ia}{4}\right)\right] = \pi\mathrm{sech}\left(\frac{\pi a}{2}\right)$$

where $\psi^{(0)}$ is the digamma function.

🗇 One can also evaluate the general form

$$\int_0^1 \left(\frac{1}{1-x} + \frac{1}{\ln x}\right)^m dx \quad m \geqslant 1$$

206 Let Li_2 denote the [dilogarithm function]. Prove that

$$\int_0^\infty \mathrm{Li}_2\left(e^{-\pi x}\right) \arctan x \, dx = \frac{\pi^2}{18} - \frac{3\zeta(3)}{8}$$

207 Prove that

$$\sum_{n=1}^\infty \arctan\left(\frac{10n}{(3n^2+2)(9n^2-1)}\right) = \ln 3 - \frac{\pi}{4}$$

208 Let ζ denote the Riemann zeta function. Prove that

$$\sum_{k=1}^\infty \frac{k\zeta(2k)}{4^{k-1}} = \frac{\pi^2}{4}$$

209 Let Li_3 denote the [trilogarithm function]. Prove that

$$\sum_{n=1}^\infty \mathrm{Li}_3\left(e^{-2n\pi}\right) = \frac{7\pi^3}{360} - \frac{\zeta(3)}{2}$$

(Seraphim Tsipelis)

210 Prove that

$$\int_0^{2-\sqrt{3}} \frac{\arctan t}{t} \, dt = \frac{\pi}{12} \log\left(2-\sqrt{3}\right) + \frac{2\mathcal{G}}{3}$$

where $\mathcal{G}$ denotes the [Catalan constant].

211 Prove that

$$\sum_{n=1}^\infty \frac{\zeta(2n+1)}{(n+1)(2n+1)} = 1 - \gamma$$

where γ stands for the [Euler - Mascheroni constant].

(Seraphim Tsipelis, Kotronis Anastasios)

212 Evaluate the following double series

$$S = \sum_{m=1}^\infty \sum_{n=1}^\infty (-1)^{m+n} \frac{m\ln(m+n)}{(m+n)^3}$$

(Enkel Hysnelaj)

213 Let $\mathcal{H}_n$ denote the n-th harmonic number. Prove that

$$\sum_{n=1}^\infty \frac{\mathcal{H}_n}{n}\left(\zeta(2) - \sum_{k=1}^n \frac{1}{k^2}\right) = \frac{7\zeta(4)}{4}$$

where ζ is the Riemann zeta function.

214 Let $\mathcal{H}_n$ denote the n-th harmonic number. Prove that

$$\sum_{n=1}^\infty \frac{\mathcal{H}_n}{n} \cos\left(\frac{n\pi}{3}\right) = -\frac{\pi^2}{36}$$

215 Prove that

$$\sum_{j=2}^\infty \prod_{k=1}^j \frac{2k}{j+k-1} = \pi$$

216 *This series may be called as "The harmony of the harmony". Evaluate the series*

$$\sum_{n=1}^\infty \frac{1}{(n+2)2^{n+2}} \sum_{k=1}^n \frac{1}{k+1} \sum_{m=1}^k \frac{1}{m} = \frac{\ln^3 2}{6}$$

217 Let $\mathbb{Z} \ni k \geqslant 1$. Prove that

$$\int_0^1 \ln^k(1-x)\ln x \, dx = (-1)^{k+1} k! \left(k+1 - \sum_{m=2}^{k+1} \zeta(k)\right)$$

where ζ denotes the Riemann zeta function.

(Ovidiu Furdui)

218 Evaluate the series

$$S = \sum_{n=1}^\infty \frac{\cos \frac{n\pi}{3}}{9-4n^2}$$

219 Let $r \in \mathbb{R}$. Prove that

$$\sum_{n=-\infty}^\infty \arctan\left(\frac{\sinh r}{\cosh n}\right) = \pi r$$

The more general identity

$$\prod_{n=-\infty}^\infty \left(1 + \frac{\sin r}{\cosh n}\right) = e^{\pi r - r^2}$$

for $\mathfrak{Re}(r) = 0$ seems to be true as pointed out by Tintarn at [AoPS.com].

(H. Ohtsuka)

228 Calculate

$$S = \sum_{n=1}^{\infty} \arctan\left(\sinh n\right) \arctan\left(\frac{\sinh 1}{\cosh n}\right)$$

220 Evaluate

$$\int_{-\infty}^{\infty} \frac{\arctan x}{x^2 + x + 1}\, dx$$

(H. Ohtsuka)

221 Let Γ denote the Gamma function. Evaluate the integral

$$\int_0^1 \left(\log \Gamma(x) + \log \Gamma(1-x) \right) \log \Gamma(x)\, dx$$

229 Let $\{\cdot\}$ denote the fractional part. Evaluate

$$\mathcal{J} = \int_0^{\pi/2} \frac{\{\tan x\}}{\tan x}\, dx$$

222 Evaluate the integrals

(i) $\displaystyle\int_0^{\infty} \frac{\ln x}{e^x + 1}\, dx$ (ii) $\displaystyle\int_0^{\infty} \frac{\ln x}{e^x - 1}\, dx$

230 Calculate

$$\mathcal{J} = \int_0^{\pi/2} x \ln \tan x\, dx$$

223 Let erf denote the error function. Prove that

$$\int_0^{\infty} e^{-x} \mathrm{erf}^2(x)\, dx = \frac{2\sqrt{2}}{\pi} \arctan \frac{1}{\sqrt{2}}$$

231 Let γ denote the Euler - Mascheroni constant. Prove that

$$\int_0^{\infty} \frac{\cos x^2 - \cos x}{x}\, dx = \frac{\gamma}{2}$$

224 Evaluate

$$\int_0^{\infty} \left(\frac{x}{e^x - e^{-x}} - \frac{1}{2} \right) \frac{dx}{x^2}$$

232 Calculate

$$\int_0^{\infty} \frac{\log x}{(2x+1)(x^2 + x + 1)}\, dx$$

225 Prove that

$$\int_0^1 \frac{\log(1+x)\log^2 x}{1-x}\, dx = \frac{7}{2}\log 2 \zeta(3) - \frac{19}{720}\pi^4$$

233 Let $\{\cdot\}$ denote the fractional part. Evaluate

$$\int_0^1 \left\{\frac{1}{x}\right\}^2 \left\{\frac{1}{1-x}\right\}\, dx$$

(Cornel Ioan Valean)

234 Let Ω denote the root of the equation $xe^x = 1$. Prove that

$$\int_{-\infty}^{\infty} \frac{dx}{(e^x - x)^2 + \pi^2} = \frac{1}{1+\Omega}$$

226 Let $\mathcal{H}_n$ denote the n-th harmonic number. Evaluate the sum

$$S = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} (-1)^{k+n} \frac{\mathcal{H}_{k+n}^2}{k+n}$$

(Cornel Ioan Valean)

235 Evaluate the series

$$S = \sum_{n=-\infty}^{\infty} \frac{x^2}{n^2 + n - 1}$$

as well as the product

227 Calculate

$$S = \sum_{n=1}^{\infty} \prod_{k=1}^{n} \frac{1 + k\log k}{2 + (k+1)\log(k+1)}$$

$$\Pi = \prod_{n=1}^{\infty} \left(1 + \frac{x^2}{n^2 + n - 1}\right)$$

236 Let ζ denote the Riemann zeta function. Prove the identity:

$$\frac{1}{2\pi}\mathrm{Li}_2\left(e^{-2\pi}\right) = \log(2\pi) - 1 - \frac{5\pi}{12} - \sum_{n=1}^{\infty}\frac{(-1)^n\zeta(2n)}{n(2n+1)}$$

where Li_2 denotes the dilogarithm function.

237 Let $\mathcal{G}$ denote the Catalan's constant. Prove that

$$\frac{\pi}{2}\sum_{n=0}^{\infty}\frac{\zeta(2n)}{(2n+1)4^n} = \mathcal{G}$$

where ζ denotes the Riemann zeta function and $\zeta(0) = -\frac{1}{2}$.

238 Let $s \in \mathbb{C}$ such that $\mathfrak{Re}(s) > 1$. Evaluate the following double Euler sum

$$S = \sum_{(j,k)\in\mathbb{Z}^2\backslash\{(0,0)\}}\frac{1}{(j^2+k^2)^s}$$

239 Evaluate the integral

$$\mathcal{J} = \int_0^{\pi/2}\sin^2 x\log\left(\sin^2(\tan x)\right)\,dx$$

240 Let $0 \leqslant \alpha, \beta \leqslant \pi$ and $\kappa > 0$. Prove that

$$\int_0^{\infty}\frac{1}{x}\log\left(\frac{x^2+2\kappa x\cos\beta+\kappa^2}{x^2+2\kappa x\cos\alpha+\kappa^2}\right)\,dx = \alpha^2 - \beta^2$$

241 Let γ denote the Euler – Mascheroni constant. Define $F(x) = \sum_{n=1}^{\infty}x^{2^n}$. Prove that

$$\gamma = 1 - \int_0^1\frac{F(x)}{1+x}\,dx$$

242 Let $\mathcal{B}_n$ denote the n-th Bernoulli number. Prove that

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\mathcal{B}_{2n}x^{2n}}{2n\,(2n)!} = \log\frac{x}{2} - \log\sin\frac{x}{2}$$

243 Evaluate the integral

$$\mathcal{J} = \int_0^1\frac{1-x}{\log x}\sum_{n=0}^{\infty}x^{2^n}\,dx$$

244 Prove that

$$\sum_{n=1}^{\infty}\frac{\coth n\pi}{n^7} = \frac{19\pi^7}{56700}$$

245 Evaluate the sum

$$S = \sum_{n=-\infty}^{\infty}\frac{\log\left|n+\frac{1}{4}\right|}{n+\frac{1}{4}}$$

(Seraphim Tsipelis)

246 Let $\mathcal{H}_n$ denote the n-th harmonic sum. Evaluate the sum:

$$S = \sum_{n=1}^{\infty}\left(\mathcal{H}_n - \log n - \gamma - \frac{1}{2n} + \frac{1}{12n^2}\right)$$

(M. Omarjee)

247 Prove that

$$\prod_{n=0}^{\infty}\left(\prod_{k=0}^{n}(k+1)^{(-1)^{k+1}\binom{n}{k}}\right)^{\frac{n(n+1)}{2n+3}} = e^{7\zeta(3)/24\zeta(2)}$$

where ζ denotes the Riemann zeta function.

248 Let $\mathbb{R} \ni s > 2$. Evaluate the (double) sum:

$$S = \sum_{(m,n)\in\mathbb{Z}^2\backslash\{(0,0)\}}\frac{m^2+4mn+n^2}{(m^2+mn+n^2)^s}$$

(Kent Merryfield)

249 Let $a \in [-\pi, \pi]$ and let us denote with Ci the Cosine integral function. Evaluate the series

$$S = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}\mathrm{Ci}(na)}{n^2}$$

🝙

🝙The most straight forward approach is to use Fourier series beginning by equation (2) at the link. The final answer is

$$S = \frac{\gamma\pi^2}{12} + \frac{\pi^2\ln a}{12} - \frac{\pi^2\ln 2}{12} - \frac{\zeta'(2)}{2} - \frac{a^2}{8}$$

where γ denotes the Euler - Mascheroni constant.

250 Let α , $\beta \in \mathbb{R}$ such that $0 < \alpha < \beta$. Prove that

(Cornel Ioan Valean)

$$\int_0^\infty \frac{\log x}{(x+\alpha)(x+\beta)}\, dx = \frac{1}{2(\beta-\alpha)}\left[\log^2\beta - \log^2\alpha\right]$$

(Grigorios Kostakos)

251 Let γ denote the Euler - Mascheroni constant. Prove that

$$\int_0^\infty \frac{\cos x^n - \cos x^{2n}}{x}\log x\, dx = \frac{12\gamma^2 - \pi^2}{2(4n)^2}$$

252 Calculate

$$\mathcal{M} = \int_0^\infty \int_0^\infty \cdots \int_0^\infty \frac{\prod\limits_{m=1}^n \cos(x_m)}{\sum\limits_{m=1}^n x_m}\, d(x_1, x_2, \ldots, x_n)$$

253 Let $\mathcal{H}_n$ denote the n -th harmonic number. Prove that $|z| < 1$ it holds that

$$\sum_{k=1}^\infty \frac{(-1)^{k-1}\mathcal{H}_{2k}}{2k+1} z^{2k+1} = \frac{\arctan z}{2}\log\left(1 + z^2\right)$$

254 Let $\mathcal{B}_n$ denote the n-th Bernoulli number. Prove that

$$\sum_{n=1}^\infty \frac{(-1)^{n-1}\mathcal{B}_{2n}x^{2n}}{2n\,(2n)!} = \log\frac{x}{2} - \log\sin\frac{x}{2}$$

255 Let $\mathcal{G}$ denote the Catalan's constant and $\mathcal{H}_n$ the n - th harmonic number. Prove that

$$\sum_{n=1}^\infty \left(\frac{\mathcal{H}_{4n-3}}{4n-3} - \frac{\mathcal{H}_{4n-2}}{4n-2}\right) = \frac{\pi^2}{64} + \frac{\pi\log 2}{32} + \frac{\mathcal{G}}{2} -$$

$$-\frac{3\log^2 2}{16} - \frac{3\pi\log 2}{32}$$

The interested reader might as well give a try the following integral

$$\mathcal{J} = \int_0^\infty \frac{\log^2 x}{(x+\alpha)(x+\beta)}\, dx$$

256 Let $\mathbb{A}$ denote the Glashier - Kinkelin constant and γ the Euler - Mascheroni constant. Prove that

$$\prod_{k=1}^\infty \prod_{n=1}^\infty \prod_{m=1}^\infty (k+n+m)^{\frac{(-1)^{k+m+n}}{k+m+n}} =$$

$$= \frac{\mathbb{A}^{3/2}}{\pi^{3/4}e^{1/8 - (7/12+\gamma)\log 2 + \frac{1}{2}\log^2 2}}$$

(Cornel Ioan Valean)

257 Prove that

$$\sum_{n=1}^\infty \frac{1}{n}\left(\frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+3} - \cdots\right)^2 =$$

$$= \frac{\pi^2\ln 2}{6} - \frac{\ln^3 2}{3} - \frac{3}{4}\zeta(3)$$

(Ovidiu Furdui)

258 Let k be a positive integer. Evaluate the multiple sum

$$\mathcal{S} = \sum_{i_1,\ldots,i_k \geqslant 1} \frac{1}{i_1\cdots i_k(i_1 + \cdots + i_k)^2}$$

(Ovidiu Furdui)

259 Evaluate

$$\int_0^\infty \int_0^\infty \frac{d(x,y)}{(e^x + e^y)^2}$$

(Ovidiu Furdui)

Currently I do not have a solution on this but the most straight forward idea is to actually try to find the number of ways n can be written as a sum of three numbers and reduce the triple product into a single one.

For $k = 1$ the sum equals $\dfrac{(k+1)!\zeta(k+2)}{2}$ whereas for $k \geqslant 2$ the sum equals

$$k!\left(\frac{k+1}{2}\zeta(k+2) - \frac{1}{2}\sum_{i=1}^{k-1}\zeta(k+1-i)\zeta(i+1)\right)$$

260 Evaluate the integral

$$\int_0^\infty \frac{e^x - 1}{e^x + 1} \ln^k \left(\frac{e^x + 1}{e^x - 1} \right) dx$$

261 Let μ denote the Möbius function. Evaluate the series

$$\mathcal{S} = \sum_{n=1}^\infty \frac{(-1)^{\mu(n)}}{n^s}$$

where $\mathfrak{Re}(s) > 1$.

262 Let $n \in \mathbb{N}$ and ζ denote the Riemann zeta function. Prove that

$$\int_0^{\pi/2} (\log \sin x)^n \, \tan x \, dx = (-1)^n \frac{n! \, \zeta(n+1)}{2^{n+1}}$$

263 Let $\mathcal{G}$ denote the Catalan's constant. Prove that

$$27 \sum_{n=0}^\infty \frac{16^n}{(2n+3)^3 \, (2n+1)^2 \, \binom{2n}{n}^2} =$$
$$= \frac{27}{2} \left(7\zeta(3) + (3 - 2\mathcal{G}) \pi - 12 \right)$$

🛢

264 Let $\mathcal{H}_n$ denote the n - th harmonic number. Prove that

$$\sum_{n=1}^\infty \frac{(-1)^{n+1} \mathcal{H}_n \mathcal{H}_{n+1}}{(n+1)^2} = \frac{\pi^4}{480}$$

265 Express in terms of dilogarithm the series

$$\mathcal{S} = \sum_{n=1}^\infty (n \operatorname{arccot} n - 1)$$

266 Let lcm denote the least common multiple. Prove that for all $s > 1$ it holds that

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{1}{\operatorname{lcm}^s(m,n)} = \frac{\zeta^3(s)}{\zeta(2s)}$$

where ζ is the Riemann zeta function.

🛢The above series was proved by Jacopo D' Aurizio , an MSE user. The series goes deeper and is actually a closed form of the hypergeometric function

$${}_4F_3 \left(1, 1, 1, \frac{3}{2}; \frac{5}{2}, \frac{5}{2}, \frac{5}{2}; 1 \right)$$

267 The n -th Fibonacci number is defined as $F_0 = 0$, $F_1 = 1$ and recursively via the relation

$$F_{n+2} = F_{n+1} + F_n \quad \text{for all} \quad n \geqslant 0$$

Prove that

$$\sum_{n=0}^\infty \arctan \left(\frac{(-1)^n}{F_{n+1} (F_n + F_{n+2})} \right) = \arctan \left(\sqrt{5} - 2 \right)$$

268 Let ζ denote the Riemann zeta function and let $\mathbb{N} \ni s \geqslant 2$. Prove that

$$\int_0^1 \operatorname{arctanh}^s(x) \, dx = \frac{2\zeta(s) \, (2^s - 2) \, \Gamma(s+1)}{4^s}$$

269 Evaluate the product

$$\Pi = \prod_{n=1}^\infty \left(1 + \frac{1}{4n} \right)^2 \left(\frac{2n+1}{2n+1+(-1)^{n-1}} \right)^{(-1)^{n-1}}$$

270 Let T_n denote the n - th triangular number. Evaluate

$$\sum_{n=1}^\infty \frac{1}{(8T_n - 3) (8T_{n+1} - 3)}$$

271 Let $\psi^{(0)}$ denote the digamma function and μ the Möbius function. Prove that

$$\sum_{n=1}^\infty \frac{\mu(n)}{n} \psi^{(0)} \left(1 + \frac{1}{n} \right) = \frac{1}{2}$$

272 Let μ denote the Möbius function. Prove that

$$\sum_{n=1}^\infty \frac{\mu(n) \log n}{n} = -1$$

273 Let gd denote the Gudermannian function. Evaluate the integral:

$$\mathcal{J} = \iint_{[0,1]^2} \frac{\operatorname{gd}(\log xy)}{1 - xy} \, d(x,y)$$

(274) Let F_n denote the n-th Fibonacci number and let $\mathcal{H}_n^{(2)}$ denote the n-th harmonic number of weight 2. Evaluate the series

$$S = \sum_{n=1}^{\infty} \frac{F_n \mathcal{H}_{n-1}^{(2)}}{n^2 \binom{2n}{n}}$$

(275) Let $\psi^{(1)}$ denote the trigamma function. Prove that

$$\sum_{n=1}^{\infty} \psi^{(1)}(n) x^n = \frac{x}{1-x} \left(\zeta(2) - \mathrm{Li}_2(x) \right)$$

In continuity, investigate for which $x \in \mathbb{R}$ does the series converge .

(276) Let $\psi^{(1)}$ denote the trigamma function. Prove that

$$\sum_{n=1}^{\infty} \frac{\psi^{(1)}(n)\psi^{(1)}(n+1)}{n^2} = \frac{\pi^6}{840} = \frac{9\zeta(6)}{8}$$

(Seraphim Tsipelis)

(277) Let Li_2 denote the dilogarithm function. Evaluate the double integral

$$\mathcal{J} = \int_0^1 \int_0^1 \frac{\log x \log y}{(1-x)(1-y)} \frac{\mathrm{Li}_2(xy)}{xy} \, d(x,y)$$

(278) Evaluate the series

$$\Omega = \sum_{n=1}^{\infty} \arctan \left(\frac{9}{9 + (3n+5)(3n+8)} \right)$$

(Dan Sitaru)

(279) Let γ denote the Euler - Mascheroni constant and $\{\cdot\}$ the fractional function. Prove that

$$\int_0^1 \{x\} \cdot \left\{ \frac{1}{1-x} \right\} \, dx = \frac{\pi^2}{12} - \gamma$$

(280) Let $\{\cdot\}$ denote the fractional function. Prove that

$$\int_1^{\infty} \frac{\{x\}}{x^5} \, dx = \frac{1}{3} - \frac{\pi^4}{360}$$

(281) Let $a \in \mathbb{R}$. Evaluate the integral

$$\mathcal{J} = \int_0^{\infty} \frac{\sin^2 ax}{x(1-e^x)} \, dx$$

(282) Let ζ denote the zeta Riemann function and Li_2 denote the dilogarithm function. Evaluate the integral

$$\int_0^1 \left[\log x \log(1-x) + \mathrm{Li}_2(x) \right] \left(\frac{\mathrm{Li}_2(x)}{x(1-x)} - \frac{\zeta(2)}{1-x} \right) \, dx$$

(283) Let $\mathcal{H}_n$ denote the n-th harmonic number. Prove that

$$\sum_{n=1}^{\infty} \frac{\mathcal{H}_n^2}{n(n+1)} = 3\zeta(3)$$

(284) Let ζ denote the Riemann zeta function. Define

$$\zeta^*(n) = \begin{cases} \zeta(n) & , \quad n > 1 \\ \gamma & , \quad n = 1 \end{cases}$$

where γ is the Euler - Mascheroni constant. Evalate the series

$$S = \sum_{n=1}^{\infty} \frac{(\zeta^*(n) - 1)\cos\left(\frac{n\pi}{3}\right)}{n}$$

(285) Let $n, m \in \mathbb{N}$. Define:

$$S_n^{(m)} = \sum_{k=0}^{n} k^m \binom{n}{k}^{-1}$$

(a) Prove that $S_n^{(1)} = \frac{n}{2} S_n^{(0)}$.

(b) Use (a) to deduce that

$$S_{n+1}^{(0)} = \frac{n+2}{2(n+1)} S_n^{(0)} + 1$$

(c) Prove that

$$\sum_{k=0}^{n} \binom{n}{k}^{-1} = \frac{n+1}{2^{n+1}} \sum_{k=1}^{n+1} \frac{2^k}{k}$$

The result is $4\zeta(2)\zeta(3) - 9\zeta(5)$.

(286) Let Li_4 denote the polylogarithm of order 4. Evaluate the integral

$$\mathcal{J} = \int_0^1 \frac{\log x \log(1-x)\mathrm{Li}_4(x)}{1-x}\,dx$$

(287) Evaluate

$$\mathcal{J} = \int_0^\infty \ln^2\left(\frac{x}{x^2+1}\right)\frac{1}{(x^2+1)^2}\,dx$$

(288) Evaluate the sum

$$\sum_{n=1}^\infty \frac{4^n}{\binom{2n}{n}(4n^2-1)}$$

(289) Let L_n denote the n-th Lucas number, defined by $L_0 = 2$, $L_1 = 1$ and for all $n \geqslant 2$

$$L_n = L_{n-1} + L_{n-2}$$

Compute the series

$$\mathcal{S} = \sum_{n=1}^\infty \arctan\left(\frac{L_{n+1}^2}{1+L_n L_{n+1}^2 L_{n+2}}\right)$$

(290) Compute the multiple integral

$$\int_{[0,1]^n}\cdots\int \frac{\sum_{k=1}^n \log(1-x_k)\prod_{k=1}^n \log(1-x_k)}{\left(\sum_{k=1}^n x_k\right)\prod_{k=1}^n x_k}\,d(x_1,\ldots,x_n)$$

(291) Let $\mathcal{H}_n$ denote the n-th harmonic number. Prove that

$$\mathcal{S} = \sum_{k=1}^\infty \sum_{n=1}^\infty \frac{\mathcal{H}_n}{kn(k+n)^3} = \frac{215}{48}\zeta(6) - 3\zeta^2(3)$$

(292) Evaluate the integral

$$\mathcal{J} = \int_0^\infty \frac{\sin^2 ax}{x(1-e^x)}\,dx$$

[Note] In fact something more general holds

$$\sum_{k=0}^n a^n b^{n-k}\binom{n}{k}^{-1} = \frac{n+1}{(a+b)\left(\frac{1}{a}+\frac{1}{b}\right)^{n+1}}\sum_{k=1}^{n+1}\frac{(a^k+b^k)\left(\frac{1}{a}+\frac{1}{b}\right)^k}{k}$$

and is a consequence of a theorem named by Mansour who proved it.

(293) Prove that

$$\int_{-\infty}^\infty \sin\left(x^2+\frac{1}{x^2}\right)\,dx = \sqrt{\frac{\pi}{2}}\left(\sin 2 + \cos 2\right)$$

(294) Let $\gcd(\cdot,\cdot)$ denote the greatest common divisor. Evaluate the sum

$$\mathcal{S} = \sum_{n=1}^\infty \frac{\gcd(n,2016)}{n^2}$$

(295) Prove that

$$\sum_{n=1}^\infty \left(\prod_{k=1}^n \cos\frac{k\pi}{n}\right) = -\frac{4}{5}$$

(296) Prove that

$$\int_0^1\int_0^1\int_0^1 \frac{d(x,y,z)}{\ln x + \ln y + \ln z} = -\frac{1}{2}$$

(297) Evaluate the double integral

$$\mathcal{J} = \int_0^\infty\int_0^\infty e^{-\frac{x^2+y^2}{2}}\sin(xy)\,dx\,dy$$

(298) Prove that $\int_0^{\pi/2} \frac{x\log(1-\sin x)}{\sin x}\,dx = -\frac{\pi^3}{8}$.

(299) Prove that

$$\int_0^\infty \left(\frac{1}{\sinh x} - \frac{1}{x}\right)\frac{x}{x^2+4\pi^2 s^2}\,dx = \frac{1}{2}\left[\psi\left(s+\frac{1}{2}\right) - \psi(s+1)\right]$$

where ψ denotes the digamma function.

(300) Let $\mathcal{S}$ denote the Sophomore's constant, namely

$$\mathcal{S} = \int_0^1 t^t\,dt = \sum_{n=1}^\infty (-1)^{n-1}n^{-n} \approx 0.7834305107$$

Prove that $\displaystyle\iint_{[0,1]^2} (xy)^{xy}\,d(x,y) = \mathcal{S}$.

[Note] An interesting question is the following integral

$$\mathcal{J} = \iiint_{[0,1]^3} (xyz)^{xyz}\,d(x,y,z)$$

Open Problems

In this section we shall present some open problems.

1. Can we cover a unit square with $\frac{1}{k} \cdot \frac{1}{k+1}$ rectangles? Here $k \in \mathbb{N}$.

2. Is the sequence $\left(\frac{3}{2}\right)^n \mod 1$ dense in the unit interval?

3. Is it true that

$$\sum_{n=0}^{\infty} \frac{1 + 14n + 76n^2 + 168n^3}{2^{20n}} \binom{2n}{n}^7 = \frac{32}{\pi^3}$$

🗄

4. *(The following is called Giuga Conjecture or Agoh-Giuga Conjecture and its origins can be traced back in 1950.)* A positive integer $p > 1$ is prime if and only if

$$\sum_{i=1}^{p-1} i^{p-1} \equiv -1 \pmod{p}$$

5. Why is it so difficult to prove that $e + \pi$ is irrational?

6. Let $\left(\frac{n}{7}\right)$ denote the Legendre symbol. Is it true that

$$\frac{24}{7\sqrt{7}} \int_{\pi/3}^{\pi/2} \log \left| \frac{\tan t + \sqrt{7}}{\tan t - \sqrt{7}} \right| \, dt = \sum_{n=1}^{\infty} \left(\frac{n}{7}\right) \frac{1}{n^2}$$

7. Is the Catalan's constant defined as

$$\mathcal{G} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}$$

irrational?

8. Let $\mathcal{H}_n$ denote the n - th Harmonic number. Is it true that for all $n \geqslant 1$ it holds that

$$\sum_{d|n} d \leqslant \mathcal{H}_n + (\log \mathcal{H}_n)e^{\mathcal{H}_n}$$

🗄

9. Let $x_0 = 2$. Is it true that the sequence $\{x_n\}_{n \in \mathbb{N}}$ defined as

$$x_{n+1} = x_n - \frac{1}{x_n}$$

is unbounded?

10. Does the series

$$\mathcal{S} = \sum_{n=1}^{\infty} \frac{1}{n^3 \sin^2 n}$$

converge?

11. Is it true that

$$\lim_{n \to +\infty} \frac{1}{n^2 \sin n} = 0$$

🗄

12. Let p_n denote the n -th prime. Is the series

$$\mathcal{S} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}n}{p_n}$$

convergent? 🗄

13. Is there a dense subset of a plane having only rational distances between its points?

14. For every odd prime is it true that one has

$$0! + 1! + \cdots + (p-1)! \not\equiv 0 \pmod{p}$$

🗄

15. *(The following is known as Littlewood's conjecture.)* For α, $\beta \in \mathbb{R}$ is it true that

$$\liminf_{n \to +\infty} (n \cdot \|n\alpha\| \cdot \|n\beta\|) = 0$$

Here $\| \cdot \|$ denotes the distance to the nearest integer.

16. What is the largest possible volume of the convex hull of a space curve having unit length?

🗄This kind of identity is amenable in principle to automatic theorem-proving methods, but (using known techniques) is out of reach of current computers. Another such formula is the Cullen's Pi Formula that can be found here.

🗄Actually Jeff Lagarias showed that this is equivalent to the Riemann hypothesis!

🗄We would expect this to tend to zero, but the proof is beyond what is currently known. It is expected that the irrationality measure of π is 2 (it is known that all but a zero-measure set of real numbers have irrationality measure 2). Therefore, it is expected that the sequence tends to 0 but currently there is no proof for that.

🗄The origin of this problem traces back to Paul Erdős .

🗄This is known as Kurepa's conjecture. A proof was claimed and published in 2004 but the claim was withdrawn in 2011.

Appendix

In this appendix we shall discuss the convergence of both the sequence $x_n = \sin(\pi\alpha^n)$ and the series $\sum_{n=1}^{\infty} \sin(\pi\alpha^n)$.

We shall begin with an exercise that already lies in this booklet. We are proving that the series

$$S = \sum_{n=1}^{\infty} \sin\left(\pi\left(2 + \sqrt{3}\right)^n\right)$$

converges.

Proof. Due to the binomial expansion we have that

$$\left(2 + \sqrt{3}\right)^n + \left(2 - \sqrt{3}\right)^n = 2\mathbb{I}_n$$

where $\mathbb{I}_n$ is an integer.

Thus,

$$
\begin{aligned}
\left|\sin\left(\pi\left(2 + \sqrt{3}\right)^n\right)\right| &= \left|\sin\left(2\pi\mathbb{I}_n - \pi\left(2 - \sqrt{3}\right)^n\right)\right| \\
&= \left|\sin\left(\pi\left(2 - \sqrt{3}\right)^n\right)\right| \\
&\leqslant \pi\left(2 - \sqrt{3}\right)^n
\end{aligned}
$$

The last is a geometric progression with common ratio λ less than 1. Thus, the original series converges absolutely.

♦

In fact , the series converges to some negative number greater than $-\frac{\pi}{1+\sqrt{3}}$. The convergence of S is not exceptional. Something deeper is at play here.

Pisot–Vijayaraghavan number

Pisot–Vijayaraghavan number, also called simply a Pisot number or a **PV number** is a real algebraic integer greater than 1 all of whose Galois conjugates are less than 1 in absolute value.

These numbers were discovered by Axel Thue in 1912 and rediscovered by G. H. Hardy in 1919 within the context of diophantine approximation. They became widely known after the publication of Charles Pisot's dissertation in 1938. They also occur in the uniqueness problem for Fourier series. Tirukkannapuram Vijayaraghavan and Raphael Salem continued their study in the 1940s. Salem numbers are a closely related set of numbers.

A characteristic property of PV numbers is that their powers approach integers at an exponential rate. Pisot proved a remarkable converse: if $\alpha > 1$ is a real number such that the sequence $\|\alpha\|^n$ measuring the distance from its consecutive powers to the nearest integer is square-summable, or ℓ^2, then α is a Pisot number.

> **Definition**
>
> An algebraic integer of degree n is a root α of an irreducible monic polynomial P of degree n with integer coefficients, its minimal polynomial. The other roots of P are called the conjugates of α. If $\alpha > 1$ but all other roots of P are real or complex numbers of absolute value less than 1, so that they lie strictly inside the circle $|z| = 1$ in the complex plane, then α is called a **Pisot number, Pisot–Vijayaraghavan number** or simply **PV number**.

For instance the golden ratio φ is a real quadratic integer that is greater than 1 while the absolute value of its conjugate $-\varphi^{-1}$ is less than 1. Thus, φ is a Pisot number and its minimal polynomial is $x^2 - x - 1$.

Elementary Properties

♦ Every integer greater than 1 is a PV number. Conversely, every rational PV number is an integer greater than 1.

♦ If α is an irrational PV number whose minimal polynomial ends in k then α is greater than $|k|$. Consequently, all PV numbers that are less than 2 are algebraic units.

♦ If α is a PV number then so are its powers α^k for all natural number exponents k.

♦ Every real algebraic number field $\mathbb{K}$ of degree n contains a PV number of degree n. This number is a field generator. The set of all PV numbers of degree n in $\mathbb{K}$ is closed under multiplication.

♦ Given an upper bound M and degree n, there are only a finite number of PV numbers of degree n that are less than M.

♦ Every PV number is a Perron number (a real algebraic number greater than one all of whose conjugates have smaller absolute value).

Diophantine properties

The main interest in PV numbers is due to the fact that their powers have a very "biased" distribution (mod 1). If α is a PV number and λ is any algebraic integer in the field $\mathbb{Q}(\alpha)$ then the sequence $\|\lambda\alpha^n\|$ where $\|\cdot\|$ denotes the distance from the real number x to the nearest integer, approaches 0 at an exponential rate. In particular, it is a square-summable sequence and its terms converge to 0.

Two converse statements are known: they characterize PV numbers among all real numbers and among the algebraic numbers (but under a weaker Diophantine assumption).

- Suppose α is a real number greater than 1 and λ is a non-zero real number such that $\sum\limits_{n=1}^{\infty} \|\lambda\alpha^n\|^2$ converges. Then α is a Pisot number and λ is an algebraic number in the field $\mathbb{Q}(\alpha)$ (Pisot's theorem).

- Suppose α be an algebraic number greater than 1 and λ is a non-zero real number such that $\|\lambda\alpha^n\| \to 0$. Then α is a Pisot number and λ is an algebraic number in the field $\mathbb{Q}(\alpha)$

Some more results

Pisot (1938) proved the fact that if θ is chosen such that there exists a $\lambda \neq 0$ for which the series $\sum\limits_{n=0}^{\infty} \sin^2(\pi\lambda\theta^n)$ converges then θ is an algebraic integer whose conjugates all (except for itself) have modulus less than 1, and λ is an algebraic integer of the field $\mathbb{K}(\theta)$.

The proof of this theorem is based on the lemma that for a Pisot number θ, there always exists a number λ such that $1 \leqslant \lambda < \theta$ and the following inequality is satisfied:

$$\sum_{n=0}^{\infty} \sin^2(\pi\lambda\theta^n) \leqslant \frac{\pi^2 (2\theta + 1)^2}{(\theta - 1)^2}$$

Back to the problem

It is known that for almost all $\alpha > 1$ (i.e except a set of Lebesgue measure 0), $\{\alpha^n\}$, the fractional part of α^n is an equidistributed sequence. A consequence of this is for almost all $\alpha > 1$, the sequence $\sin(\pi\alpha^n)$ does not converge to 0 and hence the series $\sum\limits_{n=1}^{\infty} \sin(\pi\alpha^n)$ diverges.

There are known exceptions to this. In particular, it is known that $\{\alpha^n\}$ is not equidistributed mod 1 if α is a PV number; i.e. an algebraic integer $\alpha > 1$ and all other roots of its minimal polynomials lie strictly inside the unit circle. Since the PV numbers have a very "biased" distribution (mod 1) we conclude that the series

$$S = \sum_{n=1}^{\infty} \sin(\pi\alpha^n)$$

converges whenever α is a PV number. Because $2 + \sqrt{3}$ is PV number we conclude that the series converges.

Problem 1: Discuss the convergence of the series

$$S = \sum_{n=1}^{\infty} \sin\left(\pi \left(5 + \sqrt{7}\right)^n \right)$$

Problem 2: Discuss the convergence of the sequence

$$x_n = \sin\left(\pi \left(5 + \sqrt{7}\right)^n \right)$$

References

Here is a list of references that indicate , potentially , the source of the majority of the problems or that of the appendix.

💬 International Fora

🌐 Mathematics Stack Exchange 🔒

Description: Mathematics Stack Exchange is a Q&A site that allows users to ask and answer questions. It is quite rich in interesting questions of all levels from trivial up to very challenging ones.

🌐 Art of Problem Solving 🔒

Description: Art of Problem Solving (abbrev: AoPS) is a site that is a great resource of mathematical competitions. It also has a college forum with plenty of interesting questions and answers.

🌐 mathimatikoi.org/forum ↗

Description: mathimatikoi.org (from the greek word that means mathematicians) is an English forum of university mathematics. Its main focus is in college level mathematics and some branches of Euclidean Geometry.

🌐 Integrals and Series 🔒

Description: Integrals and Series is a forum on discussion on Integrals and Series only. It has many topics on the evaluation of challenging integrals and series as well as studies on special functions.

 Note: *This site / forum is using Tapatalk and MathJaX is no longer rendering math equations. You are **strongly** adviced to use a bookmark so that it renders MathJaX. Unfortunately , this site (which once was a valuable resource of integrals and series) is useless anymore.*

💬 Local Fora

🌐 mathematica.gr ↗

Description: mathematica.gr is a greek site on mathematical discussions. It is a great resource on mathematical competitions , mathematical news, teaching technics as well as university and applied mathematics.

🅮 Other Sites

🌐 tolaso.com.gr 🔒

Description: The editor's personal site.

🏛 Institutions

- 🏠 University of Ioannina, Ioannina, Greece
- 🏠 University of Athens , Athens, Greece
- 🏠 University of Wisconsin , USA
- 🏠 University of Michigan, Michigan , USA

📖 Books / Journals

- 📕 American Mathematical Monthly
- 📕 Romanian Mathematical Monthly
- 📕 Asymmetry , Anastasios Kotronis
- 📕 Rudin W. Principals of Mathematical Analysis
- 📕 Principals of Multivariable Calculus , Giannoulis Ioannis , University of Ioannina
- 📕 Complex Analysis , Stein E.M and Shakarchi R

🅮 Other References

These other references may include facebook groups.